

# **CALCULUS II**

## **for Biotechnology**

Chapter 1: Multivariable Calculus

Chapter 2: Matrix Analysis

Chapter 3: Differential Equations

Chapter 4: Introduction to Probability

# Chapter 1

## **MULTIVARIABLE CALCULUS**

- 1.1 Functions of Several Variables
- 1.2 Partial Derivatives
- 1.3 Maxima and Minima
- 1.4 Total Differentials and Approximations
- 1.5 Double Integrals

# 1.1 Functions of Several Variables

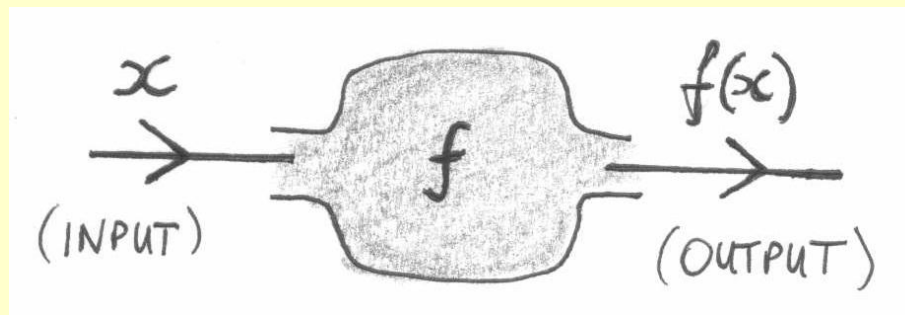
- In Calculus 1 we studied the calculus of functions of a single variable.

Examples:

- In this chapter we extend these ideas of functions of two or more variables.

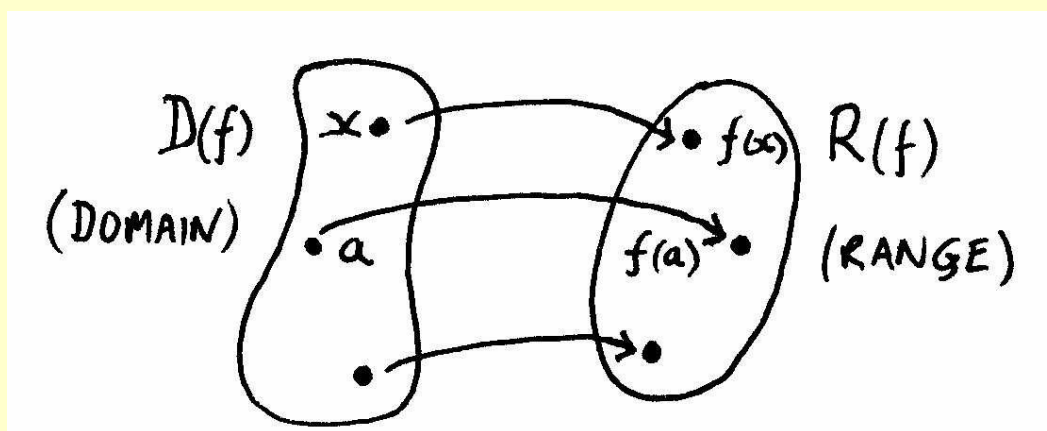
Examples:

# Functions of One Variable: Revision

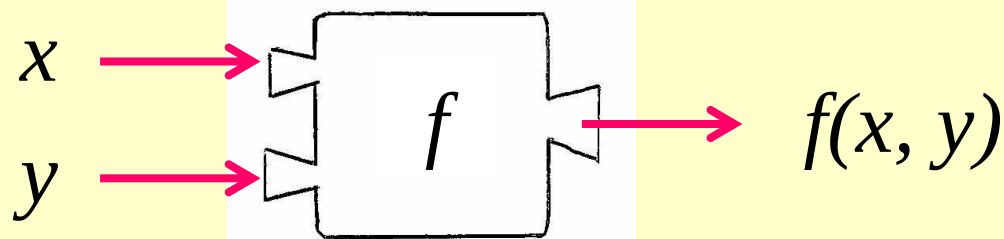


A **function**  $f$  is a rule that assigns to each element in some set  $D(f)$  exactly one element  $f(x)$  in a set  $R(f)$ . The element  $f(x)$  is called the value of  $f$  at  $x$ .

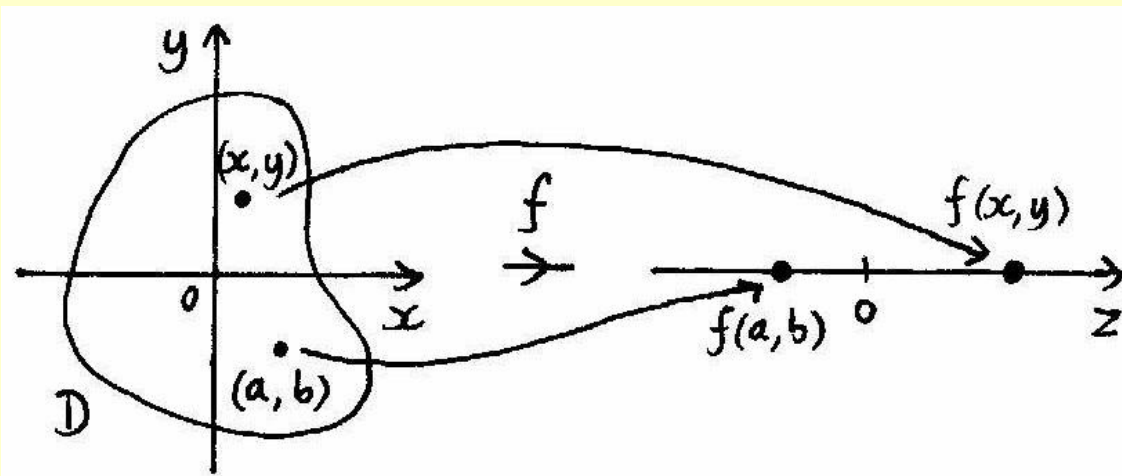
A function can also be visualized an arrow diagram:



# Functions of Two Variables



A **function**  $f$  of two variables is a rule that assigns to each ordered pair of real numbers  $(x, y)$  in a set  $D(f)$  exactly one element  $f(x, y)$  in a set  $R(f)$ .



- The domain  $D$  is a subset of the  $xy$ -plane,  $\mathbb{R}^2$
- Often we write  $z = f(x, y)$

**Example 1** For the given functions, state the domain and evaluate  $f(3,2)$

(a)  $f(x, y) = x^2 + 2y$

(b)  $f(x, y) = \frac{\sqrt{x + y + 1}}{x - 1}$

(c)  $f(x, y) = e^{x-y}$

(d)  $f(x, y) = x \ln(y^2 - x)$

## Example 2

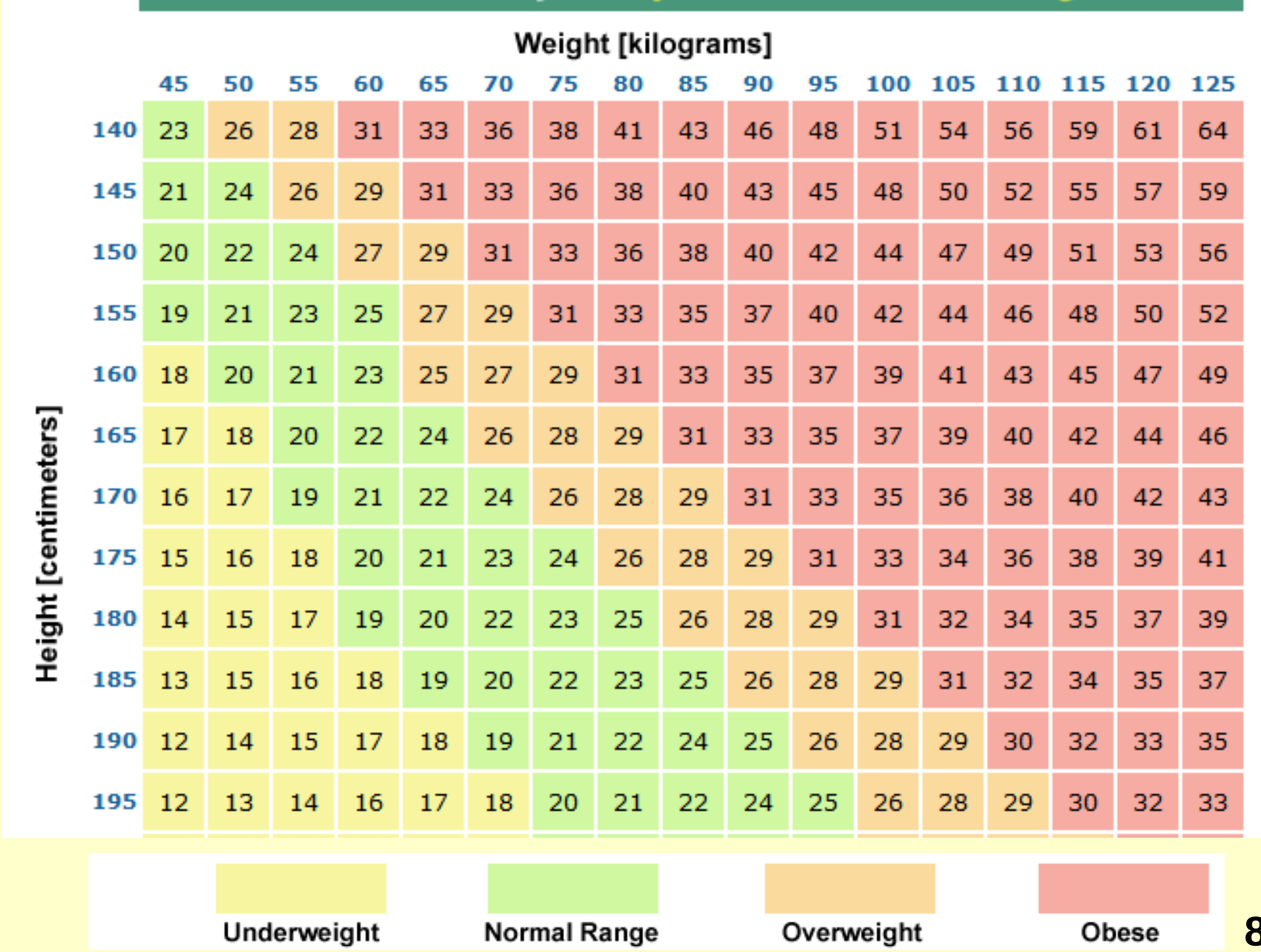
The Body Mass Index (BMI) is defined to be a person's mass (in kilograms) divided by the square of their height (in metres).

(a) Write a formula for the BMI.

(b) Calculate your personal BMI.

*Note:* For adults, a BMI of 18.5 – 25 is considered healthy!

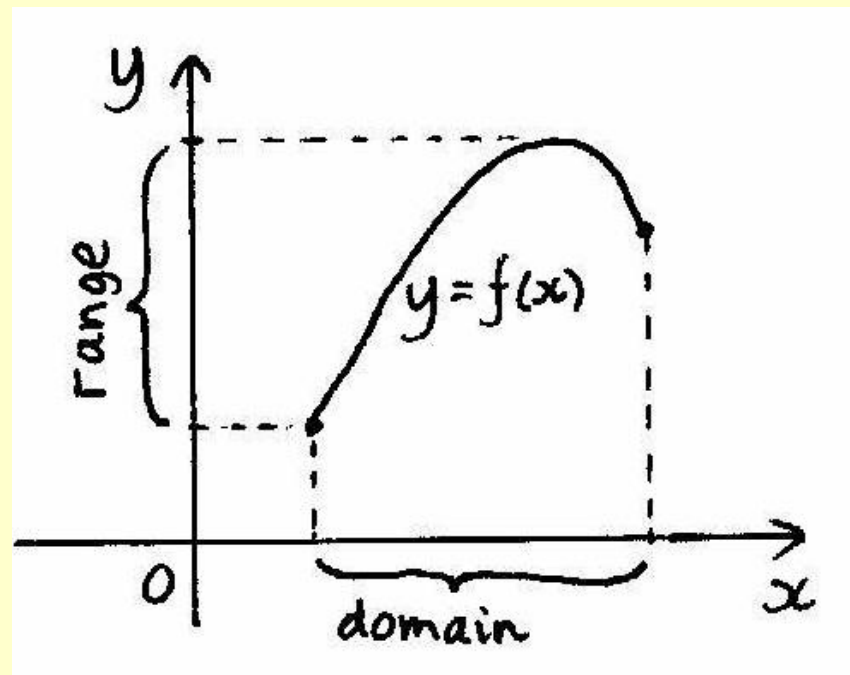
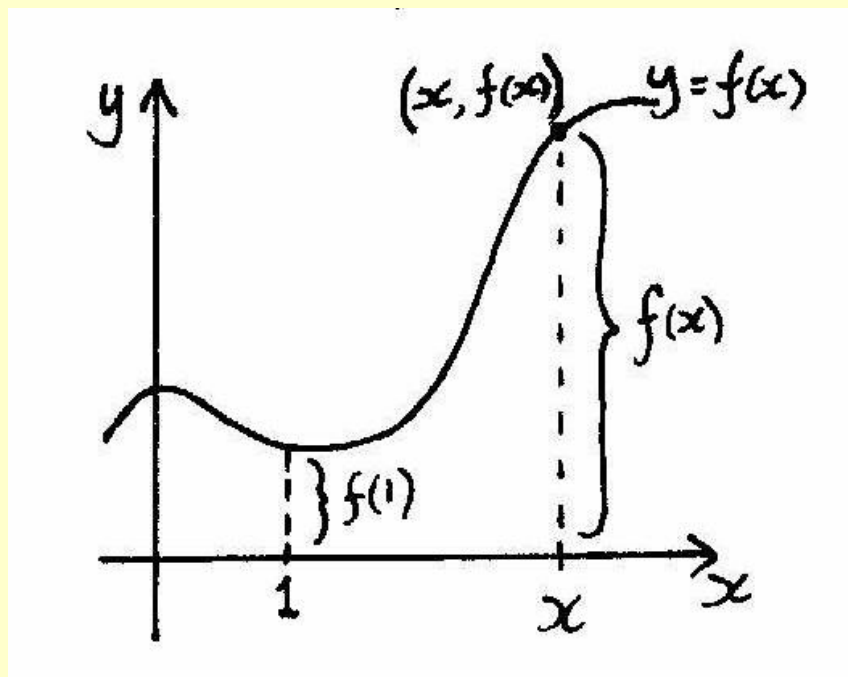
A function of two variables can be shown numerically by a chart.



# Functions of One Variable: Revision

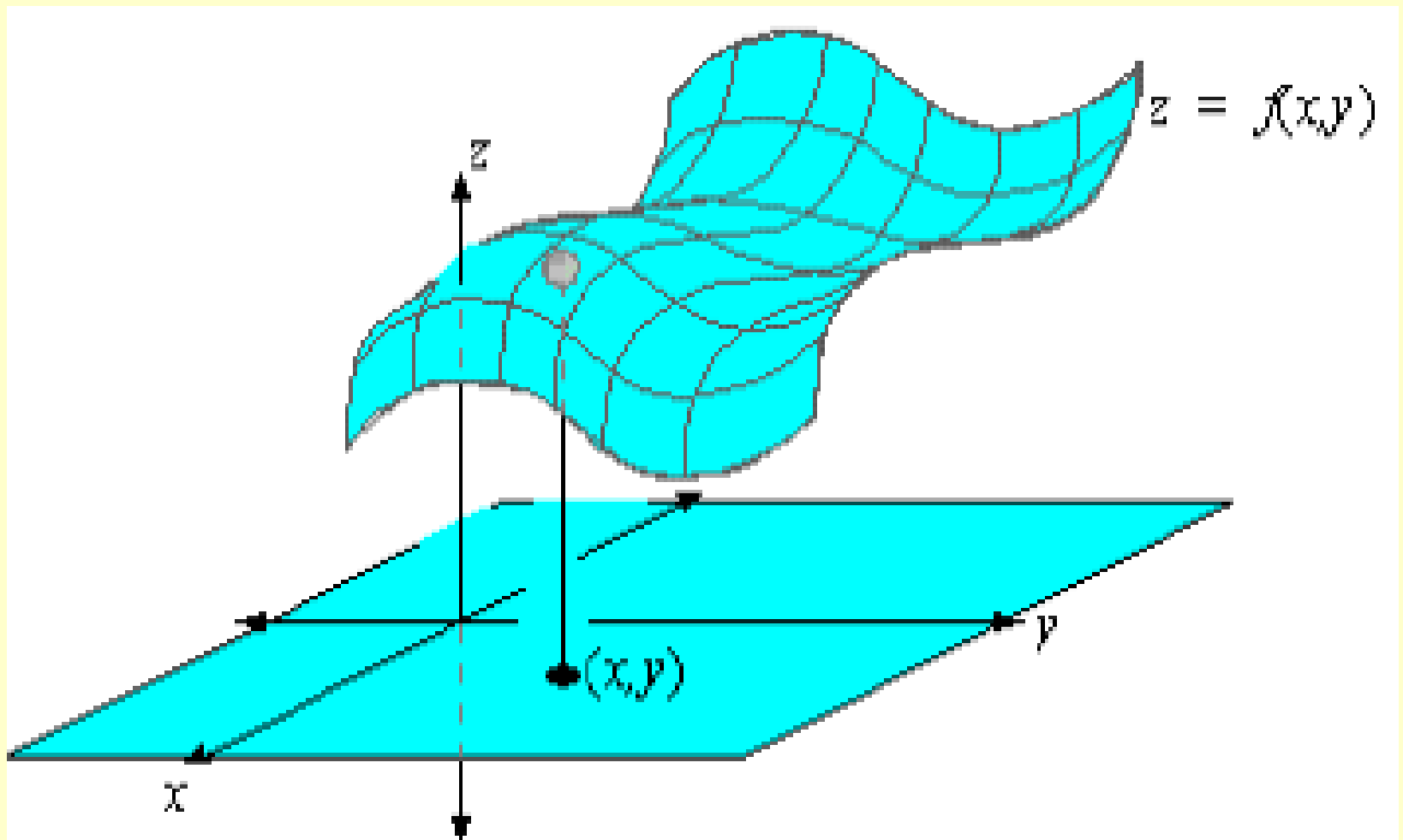
A function of one variable can be represented visually by its **graph**.

The graph of a function  $f$  is the curve  $y = f(x)$



# Functions of Two Variables

A function  $f$  of two variables can be represented by drawing the graph  $z = f(x, y)$ . This is a *surface*.



## Functions of Two Variables

- Drawing 3D surfaces is generally much more difficult than drawing 2D curves!
- Nowadays it can be done most easily using computers.
- We will look at a few simple examples of common surfaces

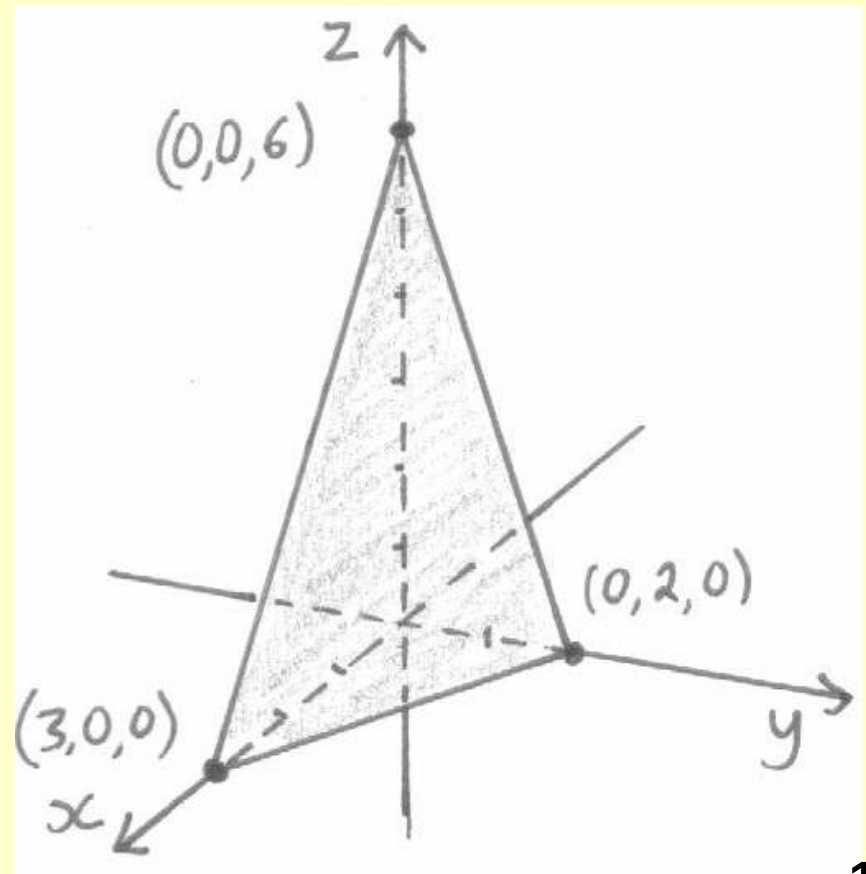
### Example 3

Describe the graph of the function  $f(x, y) = 3$

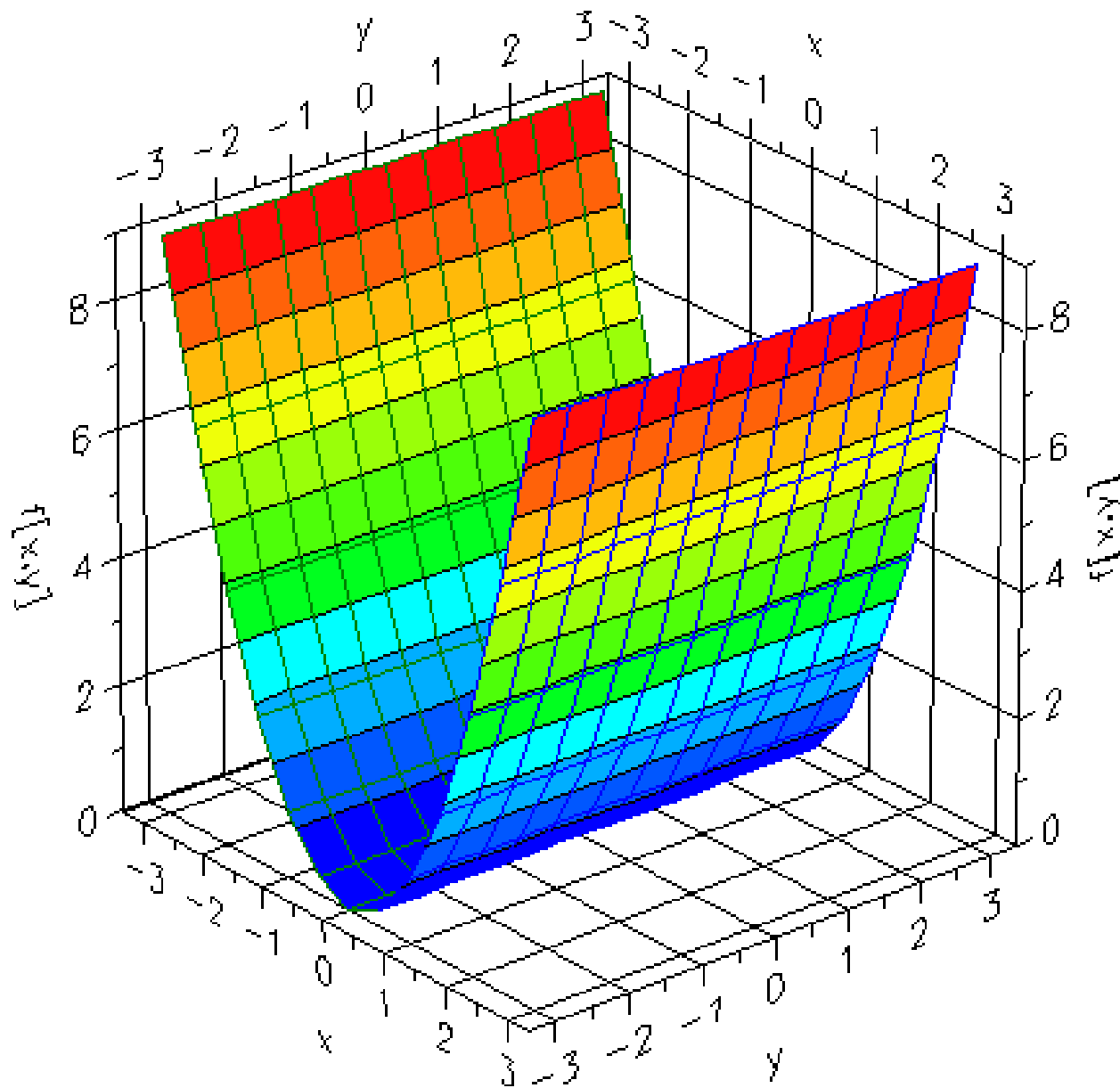
## Other Examples

(a) Consider the function  $f(x, y) = 6 - 2x - 3y$ .

- This is a linear function.
- Linear functions have form  $f(x, y) = ax + by + c$ .
- Their graphs are planes
- Planes can be sketched quite easily by finding the intercepts with the  $x$ ,  $y$  and  $z$  axes.



(b) Consider  $f(x, y) = x^2$ .



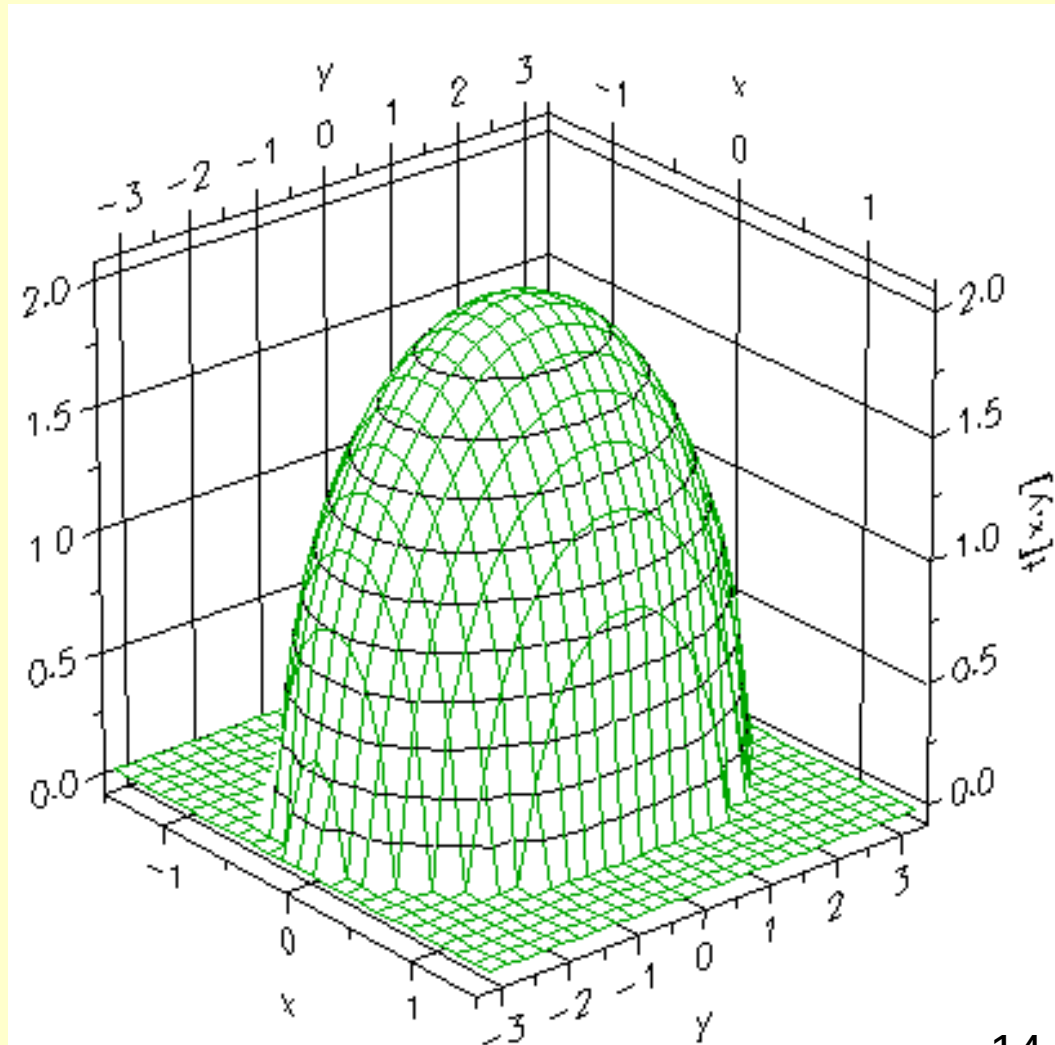
- Graph has equation  $z = x^2$ .
- $z$  does not depend on  $y$
- Any vertical plane ( $y = c$ ) intersects the surface in a curve  $z = x^2$  - called a **trace**.

(c) The surface  $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$  is an *ellipsoid*.

- It is not the graph of a function. (Why not?)
- But the top and bottom halves are graphs of functions:

$$t(x, y) = 2\sqrt{1 - x^2 - \frac{y^2}{9}}$$

$$b(x, y) = -2\sqrt{1 - x^2 - \frac{y^2}{9}}$$



It is often helpful to consider **traces**. A trace is the cross-section of the surface  $f(x, y)$  with a plane, such as  $x = 0$ ,  $x = c$ ,  $y = 0$ ,  $y = c$ ,  $z = 0$  or  $z = c$ .

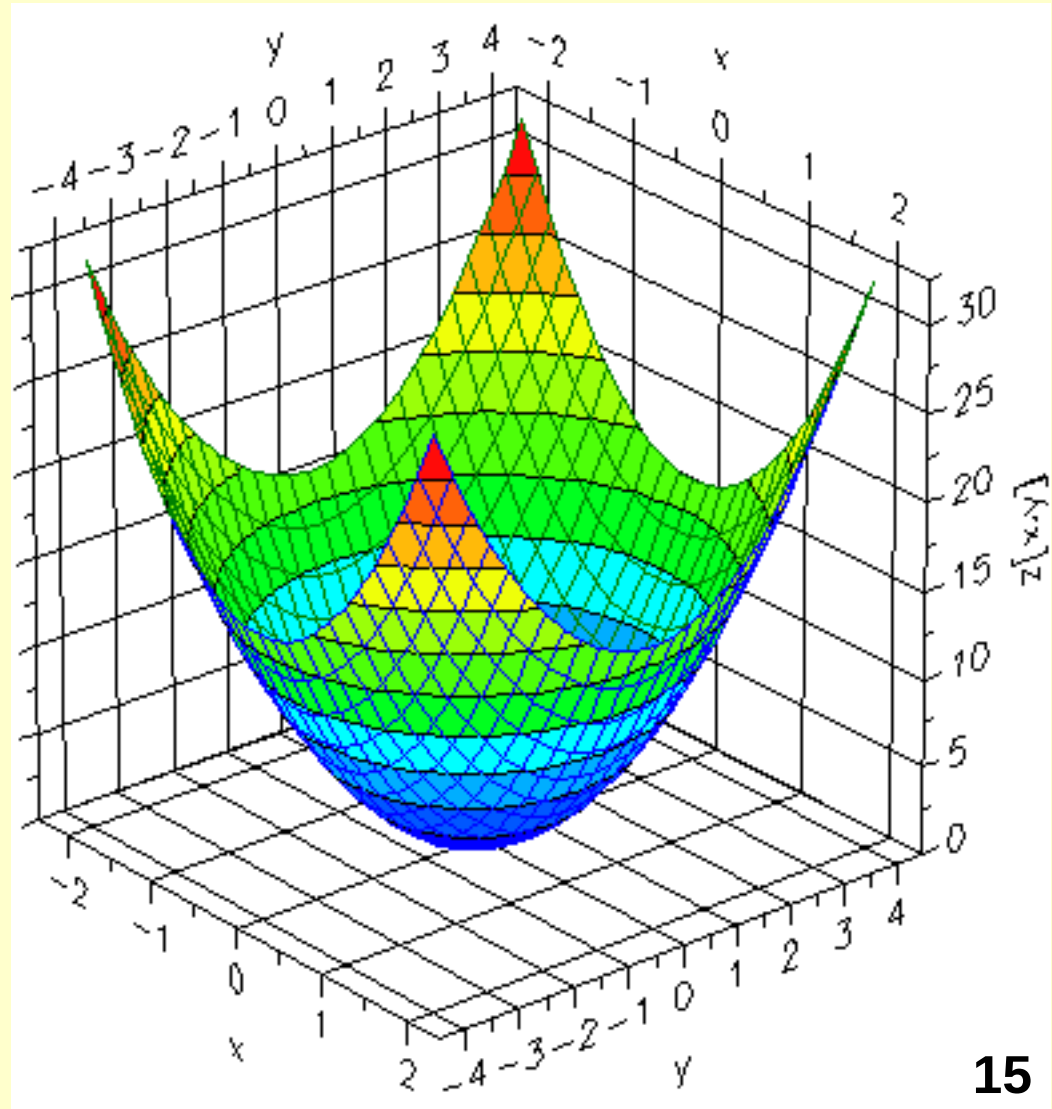
### Example (d)

$$z(x, y) = 4x^2 + y^2$$



What shape are  
vertical traces?  
horizontal traces?

Most computer-drawn  
graphs show equally-  
spaced traces.

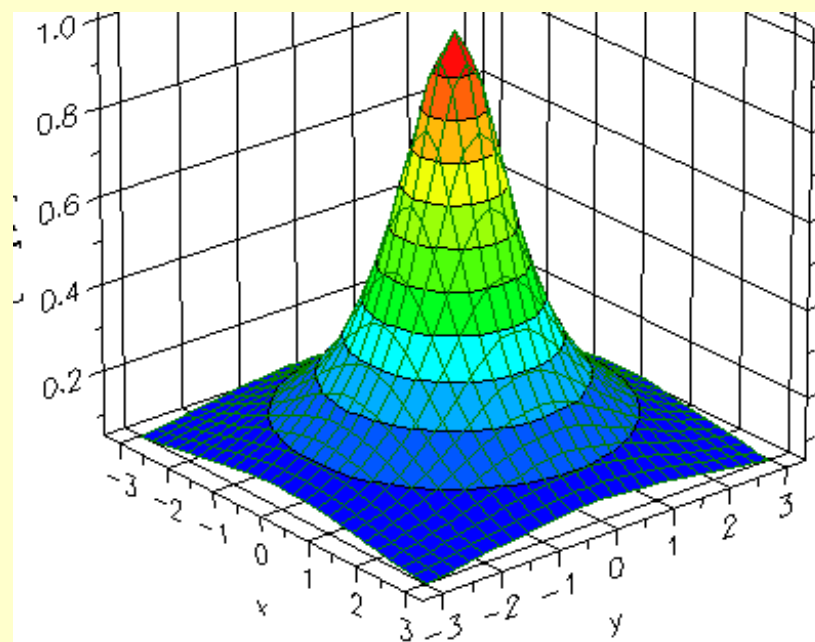
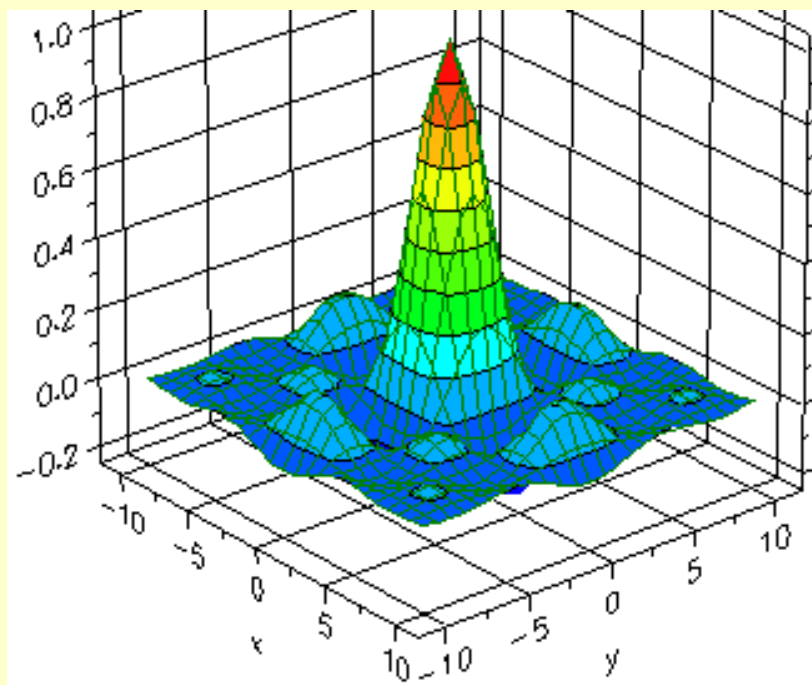
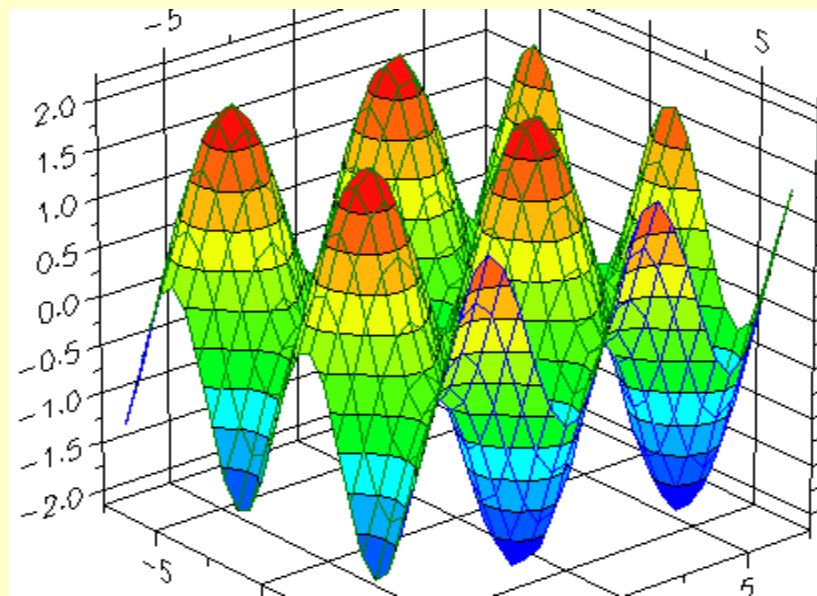


## Example 4 Which function has which graph?

$$f(x, y) = \frac{1}{1 + x^2 + y^2}$$

$$g(x, y) = \sin x + \sin y$$

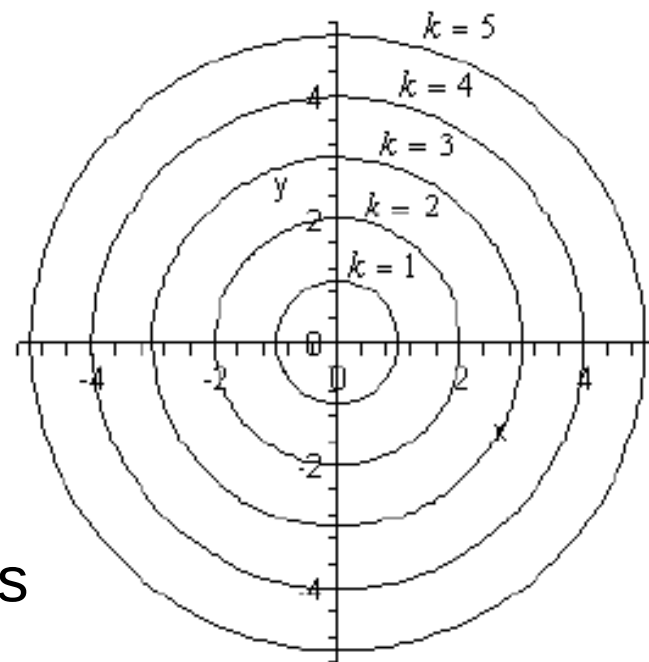
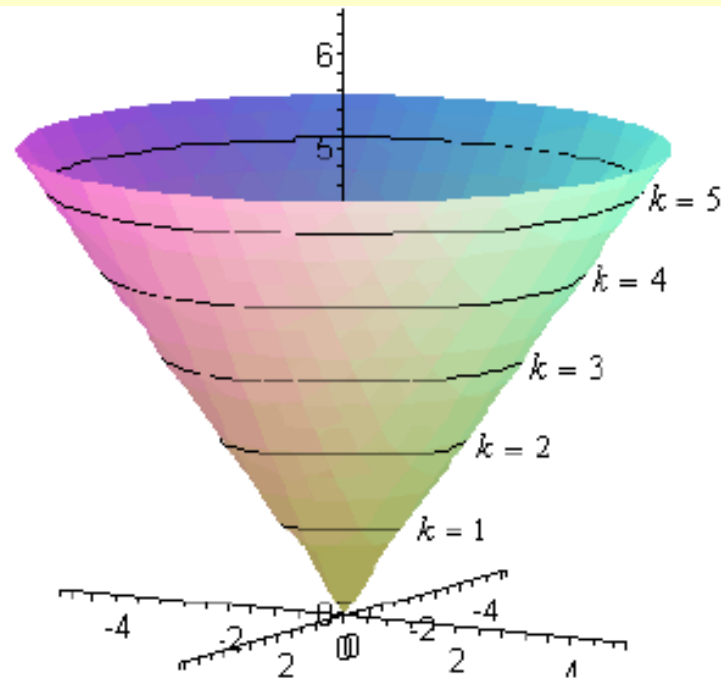
$$h(x, y) = \frac{\sin x + \sin y}{xy}$$



# Level Curves

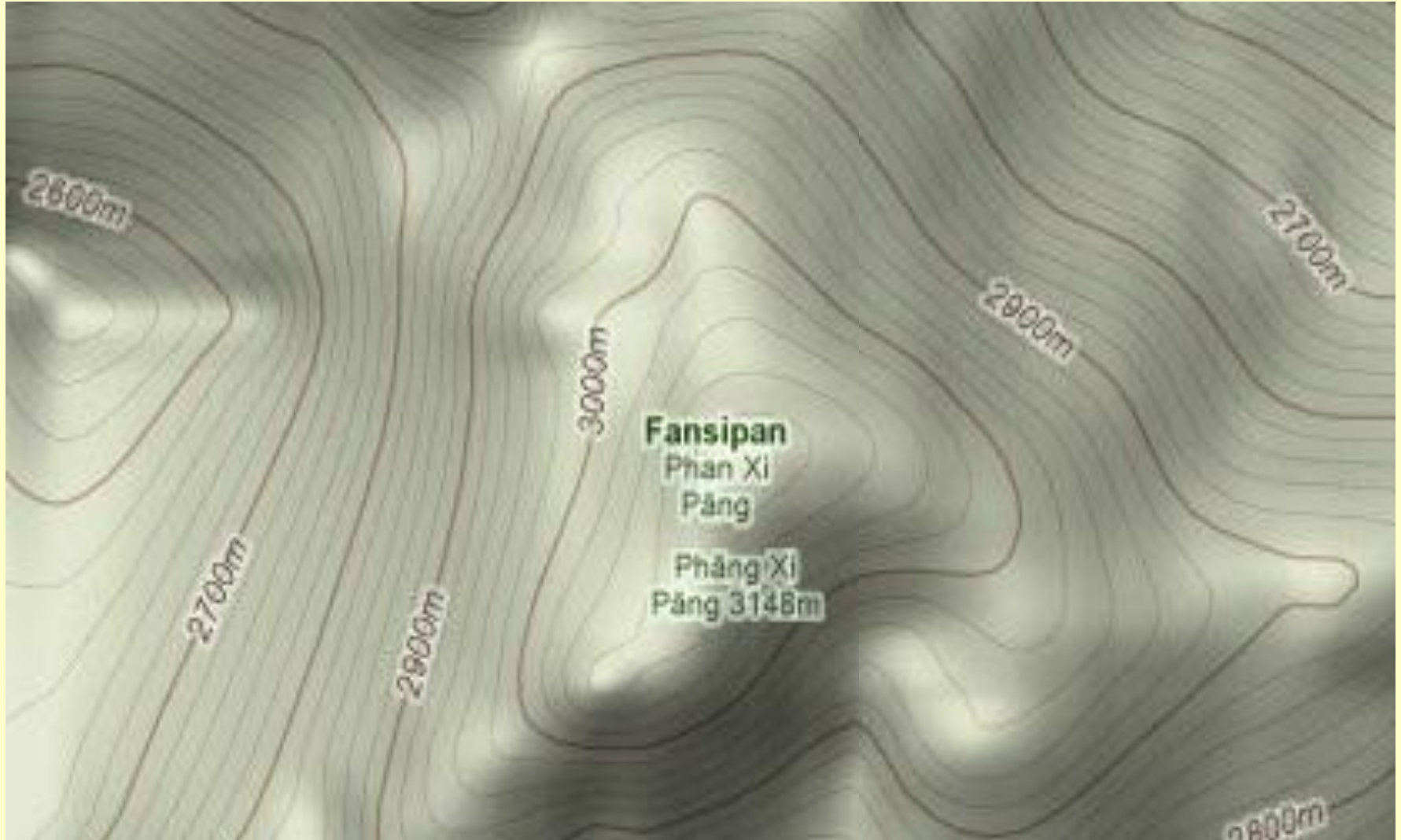
are another way of visualizing functions of two variables.

These are traces  $f(x, y) = k$  where  $k$  is a constant

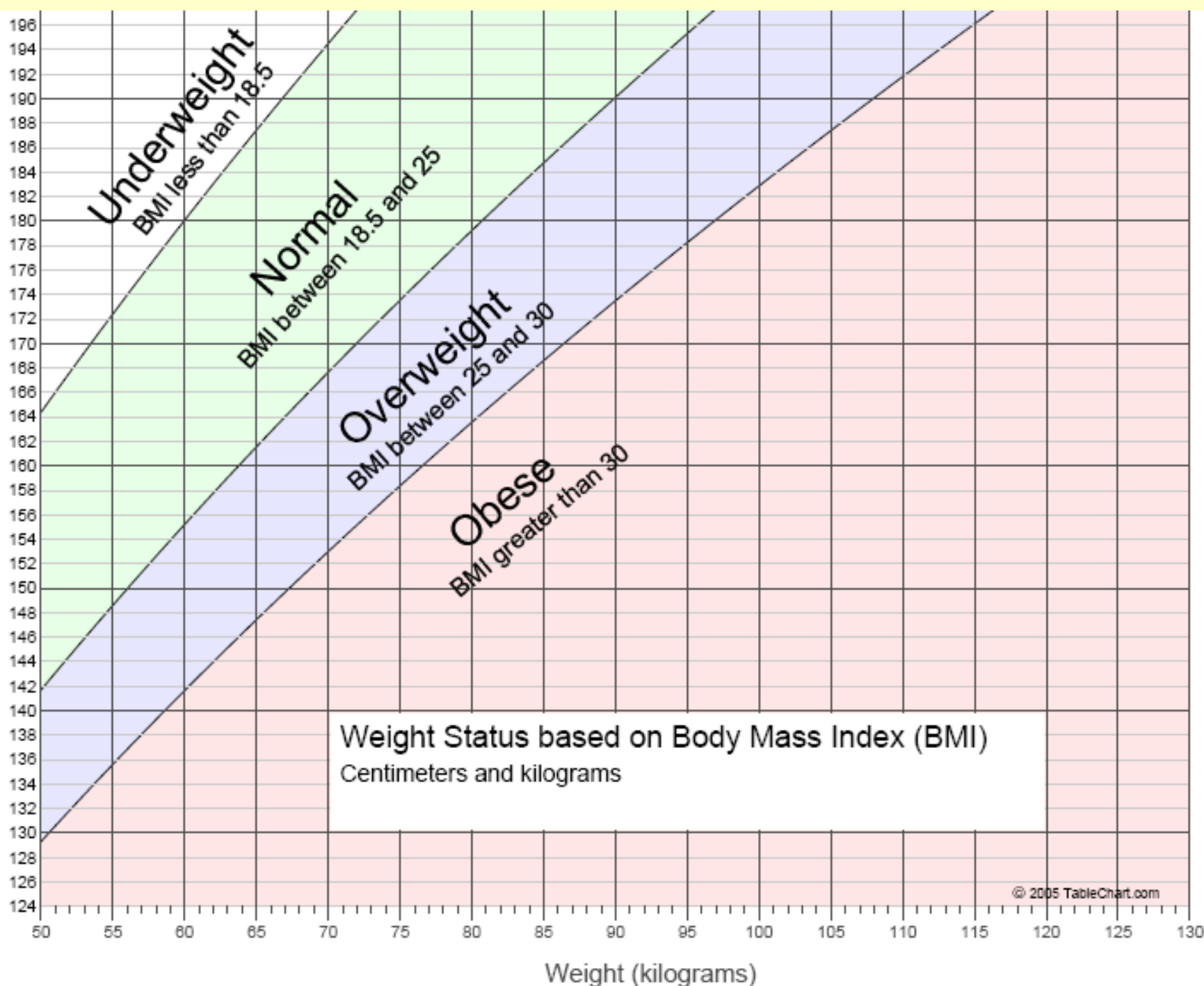


Level Curves

Level curves are also called **contour plots**.  
They are commonly seen on maps.



Height in centimeters



# Functions of Three or More variables

A function  $f$  of three variables is a rule that assigns to each ordered triple  $(x, y, z)$  in a domain  $D(f)$  exactly one element  $f(x, y, z)$  in a set  $R(f)$ .

Examples:

Similar definitions can be written for functions of more variables. However there is no simple way of visualizing such functions!

## Example 5

For the function  $f(x, y, z) = \frac{x^2}{y^2} - \cos z$ , state the domain and range and evaluate  $f(1, 2, 0)$  and  $f(0, 1, \pi)$ .

# Limits and Continuity: Revision of Functions of One Variable

- **Limits**. Remember that for a function  $f(x)$ ,

$$\lim_{x \rightarrow x_0} f(x) = L \quad \text{if and only if} \quad \lim_{x \rightarrow x_0^-} f(x) = \lim_{x \rightarrow x_0^+} f(x) = L.$$

I.e. if  $f(x) \rightarrow L$  as  $x$  approaches  $x_0$  from either above or below.

- A function  $f$  is **continuous** at  $x_0$  if  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$
- Graphically, a function is continuous where it has no hole or break – i.e. where it can be drawn without lifting one's pen off the paper!

# Limits and Continuity: Functions of Two Variables

• For a function  $f(x, y)$ , we have  $\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y) = L$  if and only if  $f(x, y) \rightarrow L$  as  $(x, y)$  approaches  $(x_0, y_0)$  along **any** path.

- But there are an infinite number of paths by which  $(x, y)$  can approach  $(x_0, y_0)$ !
- If the limit obtained by approaching along different paths is different then this proves that the limit does not exist (see Stewart 11.2 for examples).
- But how can we prove a limit *does* exist?  
Answer: using continuity ...

## Limits and Continuity, cont.

- A function  $f(x, y)$  is continuous where the surface has no holes or breaks.
- Where a function is continuous we have

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = f(x_0, y_0)$$

so the limit at that point exists and can be found

by substitution:  $L = \lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = f(x_0, y_0)$

- All the functions which are continuous with one variable remain continuous with more variables. (Just watch out for division by zero, square roots, logs of negative numbers, etc.!)

# 1.2 Partial Derivatives

In Calculus 1 we learned about the derivative of a function  $f(x)$  with respect to  $x$ .

What derivative(s) exist for a function  $f(x, y)$ ?

# Derivatives of Functions of One Variable - Revision

Consider function  $y = f(x)$ .

- The derivative of  $f(x)$  at  $a$  is: 
$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$
- The derivative function is: 
$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$
- Alternative notations: 
$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x)$$
- Interpretations of  $f'$ :
  - the *slope of the tangent* to the graph of  $f$  at  $(x, f(x))$
  - the instantaneous *rate of change* of  $f(x)$  with respect to  $x$

Now consider a function  $z = f(x, y)$ .

We want to retain the idea that a derivative is a *rate of change* ... but now we have *two* variables  $x$  and  $y$  which can change independently, providing an infinite number of ways in which  $f$  can change ... !  
Changing only  $x$  or changing only  $y$  or changing both will give different changes in  $f$  ! So what do we do??

Our approach is to change only *one variable at a time*. I.e. we vary  $x$  while keeping  $y$  constant,  
or vary  $y$  while keeping  $x$  constant.

Consider the function  $z = f(x, y) = 2x^2y^3$ .

We will look *near* the point  $(a, b)$ , where  $a, b$  are constants. We will keep  $y$  constant and let  $x$  vary.

Setting  $y = b$  gives  $f(x, b) = 2x^2b^3$

Now this is really just a function of one variable:  $x$ .

We can write:  $g(x) = f(x, b) = 2x^2b^3$

And differentiate:  $\frac{dg}{dx} = 4xb^3$

At the point  $(a, b)$ , we get  $\frac{dg}{dx} = 4ab^3$

This called ***the partial derivative of  $f$  with respect to  $x$*** . It is commonly notated  $f_x(a, b) = 4ab^3$

We can find *the partial derivative of  $f$  with respect to  $y$*  in a similar way.

First set  $x = a$ :  $f(a, y) = 2a^2 y^3 = h(y)$

Then differentiate with respect to  $y$ :  $\frac{dh}{dy} = 6a^2 y^2$

So at the point  $(a, b)$ , we have  $f_y(a, b) = 6a^2 b^2$

In fact  $(a, b)$  can be any  $(x, y)$ , so we can write partial derivative **functions**:

$$f_x(x, y) = 4xy^3$$

$$f_y(x, y) = 6x^2 y^2$$

# Partial Derivatives: Definition

The partial derivatives of a function  $f(x, y)$  are:

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

**Alternative notations:** For  $z = f(x, y)$

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x}$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y}$$

Read "partial  
d z by d x"

## Finding Partial Derivatives of $f(x, y)$

- To find  $f_x$  - regard  $y$  as a constant  
- differentiate  $f(x, y)$  with respect to  $x$
- To find  $f_y$  - regard  $x$  as a constant  
- differentiate  $f(x, y)$  with respect to  $y$

### Example 6

Given  $f(x, y) = x^3 + x^2y - y^2$  find  $f_x$ ,  $f_y$ ,  $f_x(2,1)$  and  $f_y(2,1)$

### Example 7

Given  $f(x, y) = x^4 y + 6\sqrt{y} - 1$ ,

evaluate  $\partial f / \partial x$  and  $\partial f / \partial y$  at the point  $(1, 4)$ .

### **Example 8**

Find the (first order) partial derivatives of the given functions.

$$(a) \quad f(x, y) = 2 \sin x + y^2 \cos 3x$$

$$(b) \quad g(u, v) = u^2 \ln v + \frac{u}{v^2}$$

$$(c) \quad z(x, y) = 2 \cos\left(\frac{3}{x}\right) e^{y^2}$$

$$(d) \quad h(s, t) = \frac{t^2}{s^3 t - 3s}$$

$$(e) \quad z = \sqrt{x^2 + xy + y^2}$$

# Partial Derivatives of Functions of Three or More Variables

These are defined in a similar way.

E.g. for a function  $f(x, y, z)$  we have

$$f_x(x, y, z) = \lim_{h \rightarrow 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

- I.e. to find  $f_x$
- regard  $y$  and  $z$  as constants
  - differentiate  $f(x, y, z)$  with respect to  $x$

## Example 9

Find all the (first order) partial derivatives of

$$T(x, y, z) = xe^{xy} \ln \frac{1}{z}$$

# Interpretations of Partial Derivatives:

## (I) Rates of Change

For a function  $z = f(x, y)$ ,

- $f_x$  represents the instantaneous rate of change of  $z$  with respect to  $x$ , as  $y$  is held fixed
- $f_y$  represents the instantaneous rate of change of  $z$  with respect to  $y$ , as  $x$  is held fixed

## Example 10

Consider function  $z = f(x, y) = \frac{x^2}{y^3}$  near point  $(2, 5)$ .

- Find  $f_x(2, 5)$  and  $f_y(2, 5)$ .
- Is  $z$  increasing or decreasing if we
  - (i) vary  $x$  and hold  $y$  fixed?
  - (ii) vary  $y$  and hold  $x$  fixed?
- In which case does the function change more rapidly?

**Example 11** A person of mass  $m$  kilograms and height  $h$  metres has BMI  $B(m, h) = \frac{m}{h^2}$  .

Evaluate  $B_m(55, 1.70)$  and  $B_h(55, 1.70)$  and explain their significance.

**Example 13** How hot we *feel* depends not only on the air temperature but also on the humidity. Let  $T$  be the actual temperature (in  $^{\circ}\text{C}$ ) and  $h$  the relative humidity (%). The **perceived temperature** is  $I(T, h)$ . From the chart, *estimate* the value of  $I_h(32, 80)$  and explain its meaning.

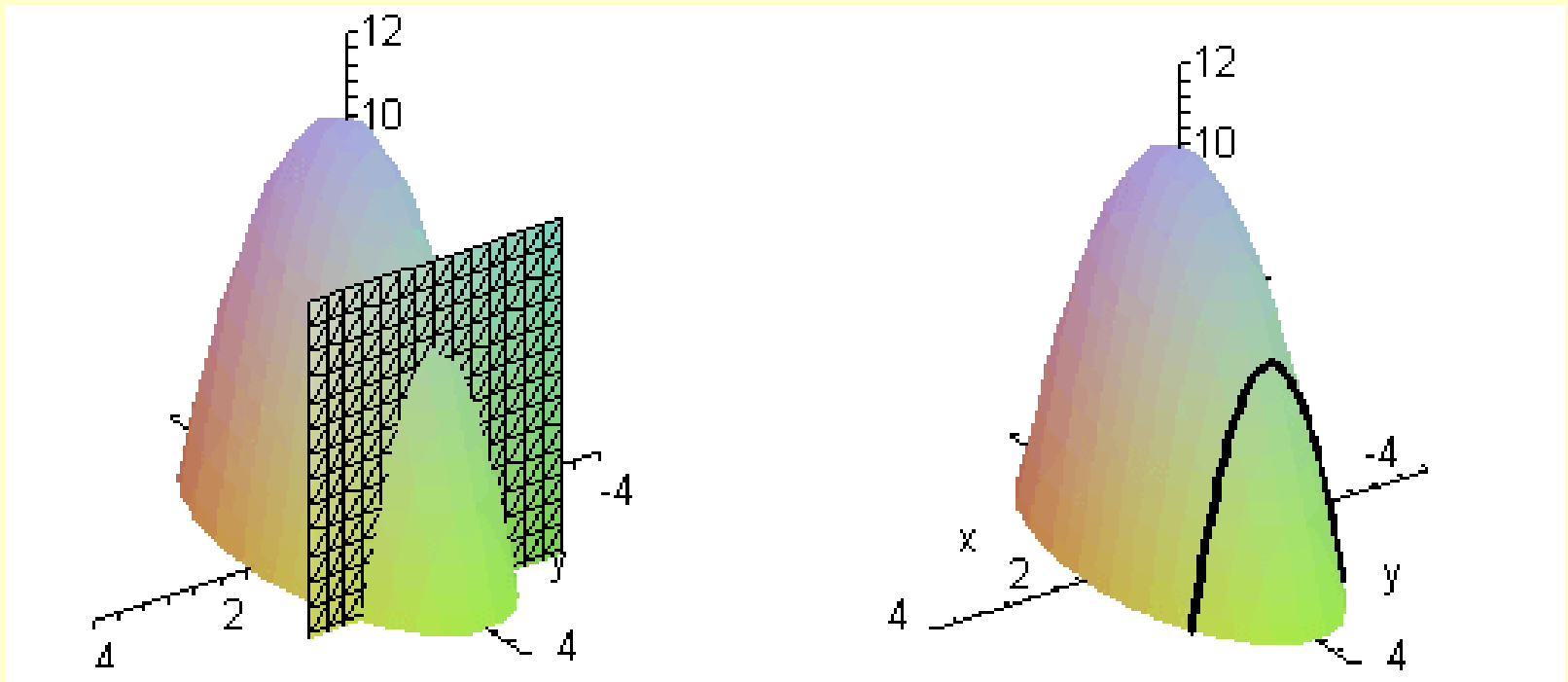
| Temperature        |  | Relative Humidity (%) |      |      |      |      |      |      |      |      |      |      |
|--------------------|--|-----------------------|------|------|------|------|------|------|------|------|------|------|
| $^{\circ}\text{C}$ |  | 100                   | 90   | 80   | 70   | 60   | 50   | 40   | 30   | 20   | 10   | 5    |
| 15                 |  | 17.5                  | 17.0 | 16.5 | 16.0 | 15.5 | 15   | 14.5 | 14   | 13.5 | 12.9 | 12.7 |
| 18                 |  | 20.2                  | 19.6 | 19.0 | 18.4 | 17.8 | 17.2 | 16.6 | 16.0 | 15.4 | 14.8 | 14.5 |
| 20                 |  | 22.2                  | 21.5 | 20.8 | 20.1 | 19.4 | 18.7 | 18.0 | 17.4 | 16.7 | 16.0 | 15.6 |
| 22                 |  | 24.2                  | 23.4 | 22.6 | 21.9 | 21.1 | 20.3 | 19.5 | 18.7 | 18.0 | 17.2 | 16.8 |
| 25                 |  | 27.5                  | 26.5 | 25.6 | 24.7 | 23.7 | 22.8 | 21.9 | 20.9 | 20.0 | 19.1 | 18.6 |
| 28                 |  | 31.0                  | 29.9 | 28.7 | 27.6 | 26.5 | 25.4 | 24.3 | 23.2 | 22.0 | 20.9 | 20.4 |
| 30                 |  | 33.5                  | 32.2 | 31.0 | 29.7 | 28.5 | 27.2 | 26.0 | 24.7 | 23.5 | 22.2 | 21.6 |
| 32                 |  | 36.1                  | 34.7 | 33.3 | 31.9 | 30.5 | 29.1 | 27.7 | 26.3 | 24.9 | 23.5 | 22.8 |
| 35                 |  | 40.4                  | 38.7 | 37.1 | 35.4 | 33.7 | 32.1 | 30.4 | 28.8 | 27.1 | 25.4 | 24.8 |
| 38                 |  | 45.1                  | 43.1 | 41.1 | 39.2 | 37.2 | 35.3 | 33.3 | 31.4 | 29.4 | 27.4 | 26.5 |
| 40                 |  | 48.4                  | 46.2 | 44.4 | 41.9 | 39.7 | 37.5 | 35.3 | 33.2 | 31.0 | 28.8 | 27.7 |

## Interpretations of Partial Derivatives: (II) Graphical

For a surface  $z = f(x, y)$ , what does  $f_x(a, b)$  represent?

- We fix  $y = b$  and consider  $f(x, b) = g(x)$

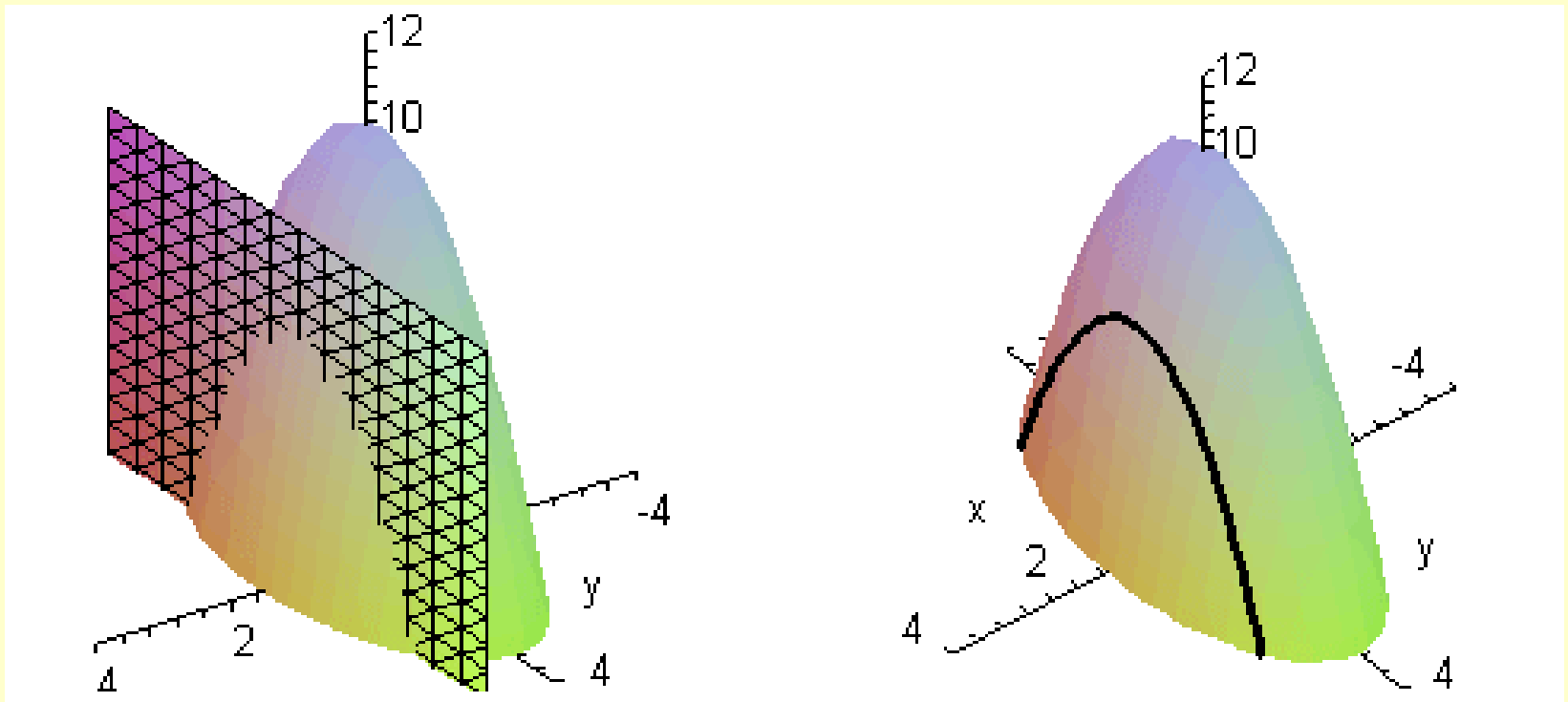
⇒ This is a curve which is a **trace** on the surface  $f$ .



- Then we find  $\frac{dg}{dx} = f_x(x, b)$  and evaluate  $f_x(a, b)$

⇒ This is the slope of the tangent to the trace

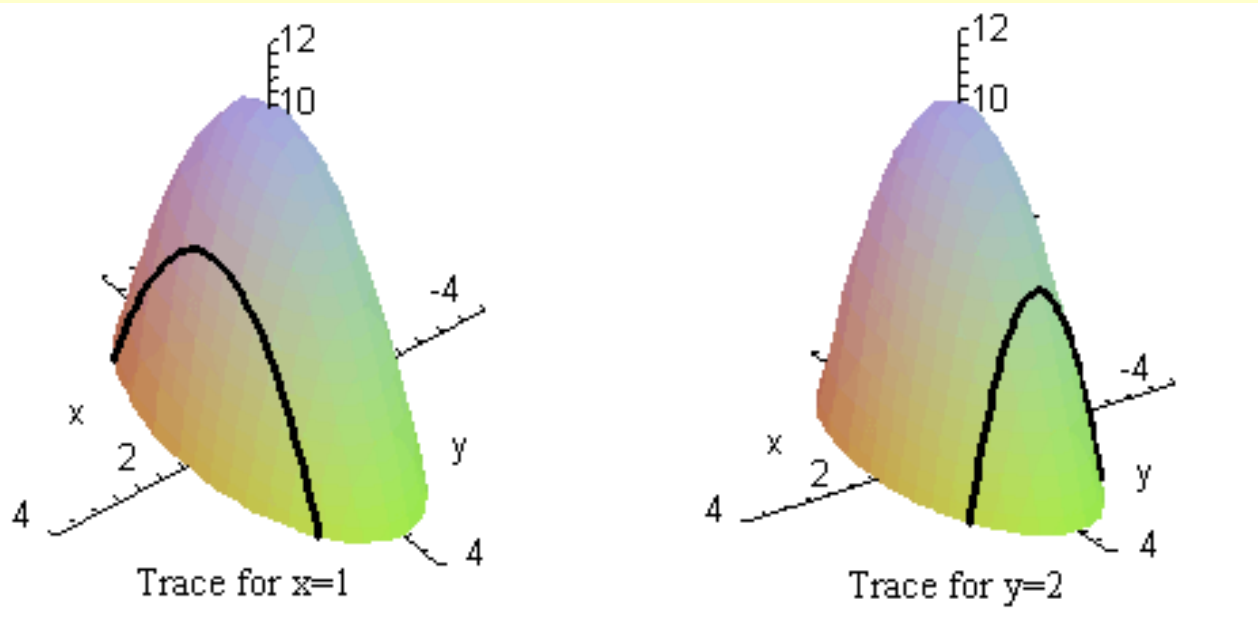
- ⇒ • Partial derivatives represent ***slopes of traces***
- $f_x(a, b)$  is the slope of the tangent to the trace of  $z = f(x, y)$  in the plane  $y = b$  at the point  $(a, b)$ .
  - $f_y(a, b)$  is the slope of the tangent to the trace of  $z = f(x, y)$  in the plane  $x = a$  at the point  $(a, b)$ .



## Example 12

Consider the surface  $z = f(x, y) = 10 - 4x^2 - y^2$ .

Find the slope of the tangent to the trace at the point  $(1, 2)$  in the plane  $y = 2$ .



# Implicit Partial Differentiation

## Implicit Differentiation - Revision

**Example** Given  $x^2 + xy^3 = ye^x$ , find  $dy/dx$

Summary:

- Differentiate each term with respect to  $x$ 
  - For functions of  $y$  use chain rule:  $\frac{d}{dx}[f(y)] = f'(y)\frac{dy}{dx}$
- Rearrange to form  $dy/dx = \dots$

# Implicit Partial Differentiation

We use a similar method.

E.g. Suppose  $z$  is defined implicitly as a function of  $x$  and  $y$ . To find  $\frac{\partial z}{\partial x}$  :

1. Differentiate each term w.r.t.  $x$ 
  - Treat  $y$  as a constant
  - For functions of  $z$  use chain rule:

$$\frac{\partial}{\partial x} [f(z)] = f'(z) \frac{\partial z}{\partial x}$$

2. Rearrange

**Example 14** A function  $z(x, y)$  is defined implicitly by  
 $x^3z^2 - 5xy^4z = x^2 + y^3$ . Find  $\partial z/\partial x$  and  $\partial z/\partial y$ .

# Second Order Partial Derivatives

Consider  $z = f(x, y)$ .

$f_x$  and  $f_y$  are both functions of  $x$  and  $y$ . So we can differentiate these again, with respect to  $x$  or  $y$ .

This gives us 4 second order derivatives:

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x}$$

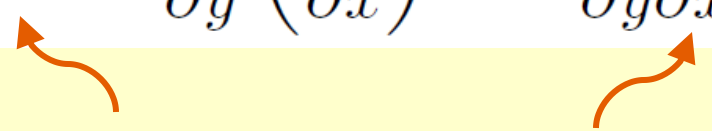
$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2}$$

# Notes

1. Derivatives such as  $\frac{\partial^2 z}{\partial x \partial y}$  are called *mixed derivatives*

2. Note the order!

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x}$$


Here we differentiate w.r.t.  $x$  first

## Example 15

Given  $z = e^{-2x} \sin(3y)$  find  $\frac{\partial^2 z}{\partial y \partial x}$ .

## Example 16

Find all the second partial derivatives of

$$z = f(x, y) = 3x^2 - x^2 e^{5y} + \cos(2y)$$

# Clairaut's Theorem

In the previous example we found

*This is not a coincidence!*

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

## Theorem

Suppose  $z = f(x, y)$  is defined on a disc  $D$  that contains the point  $(a, b)$ . If the functions  $f_{xy}$  and  $f_{yx}$  are continuous on  $D$  then  $f_{xy}(a, b) = f_{yx}(a, b)$ .

# Higher Order Partial Derivatives

Third and higher order partial derivatives can be defined in a similar way. For example:

$$f_{xyx} = (f_{xy})_x = \frac{\partial}{\partial x} \left( \frac{\partial^2 f}{\partial y \partial x} \right) = \frac{\partial^3 f}{\partial x \partial y \partial x}$$

$$f_{yxx} = (f_{yx})_x = \frac{\partial}{\partial x} \left( \frac{\partial^2 f}{\partial x \partial y} \right) = \frac{\partial^3 f}{\partial x^2 \partial y}$$

An extension to Clairaut's Theorem states that, if all continuous,  $f_{xxy}(a, b) = f_{xyx}(a, b) = f_{yxx}(a, b)$ , etc.

## Example 17

Given  $f(x, y, z) = \sin(3x + yz)$ , find  $\frac{\partial^4 f}{\partial z \partial y \partial x^2}$  .

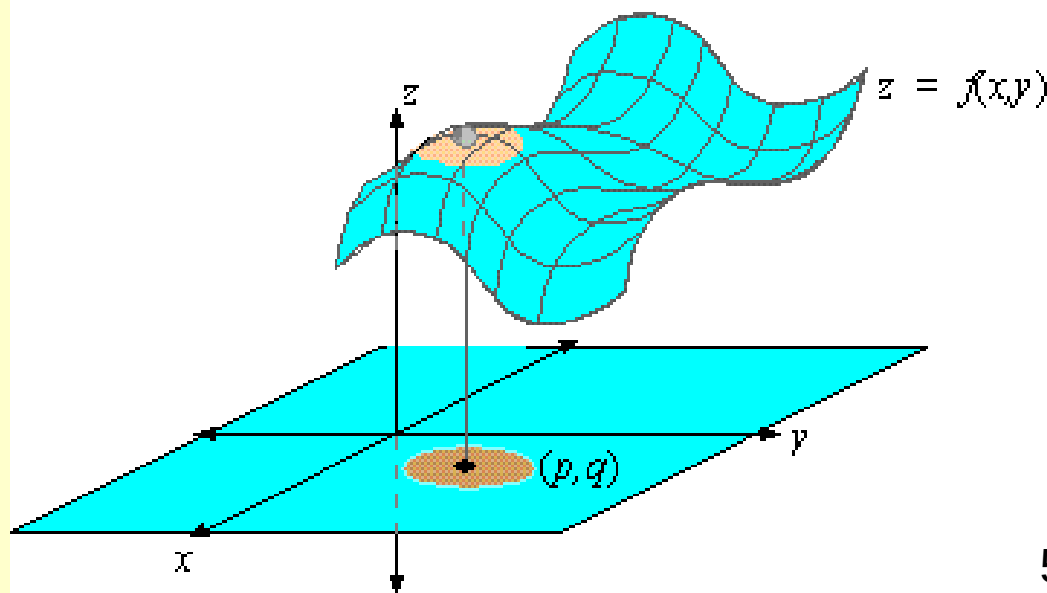
## 1.3 Maxima & Minima

- In Calculus 1, we learnt how to find and identify maxima and minima (generally known as “extrema”) for a function  $y = f(x)$ .
- Here we will do similar work for functions  $z = f(x, y)$ .
- Remember that there are two types of extrema: *local* extrema and *absolute* extrema. We will look at local extrema first, then absolute extrema.

# Relative Maxima and Minima: Definition

A function  $f(x, y)$  has a **local maximum** (or **relative maximum**) at  $(a, b)$  if  $f(a, b) \geq f(x, y)$  for all points  $(x, y)$  in some region around  $(a, b)$ .

A function  $f(x, y)$  has a **local minimum** (or **relative minimum**) at  $(a, b)$  if  $f(a, b) \leq f(x, y)$  for all points  $(x, y)$  in some region around  $(a, b)$ .



# Functions of One Variable: Revision

Remember that for a function  $y = f(x)$ ,

- A critical point is a point where  $f'(x) = 0$  or  $f'(x)$  does not exist
- If  $x = a$  is a local extremum then it is a *critical point*

*Remember* : Not all critical points are local extrema,  
but all local extrema are critical points!

So to find local extrema:

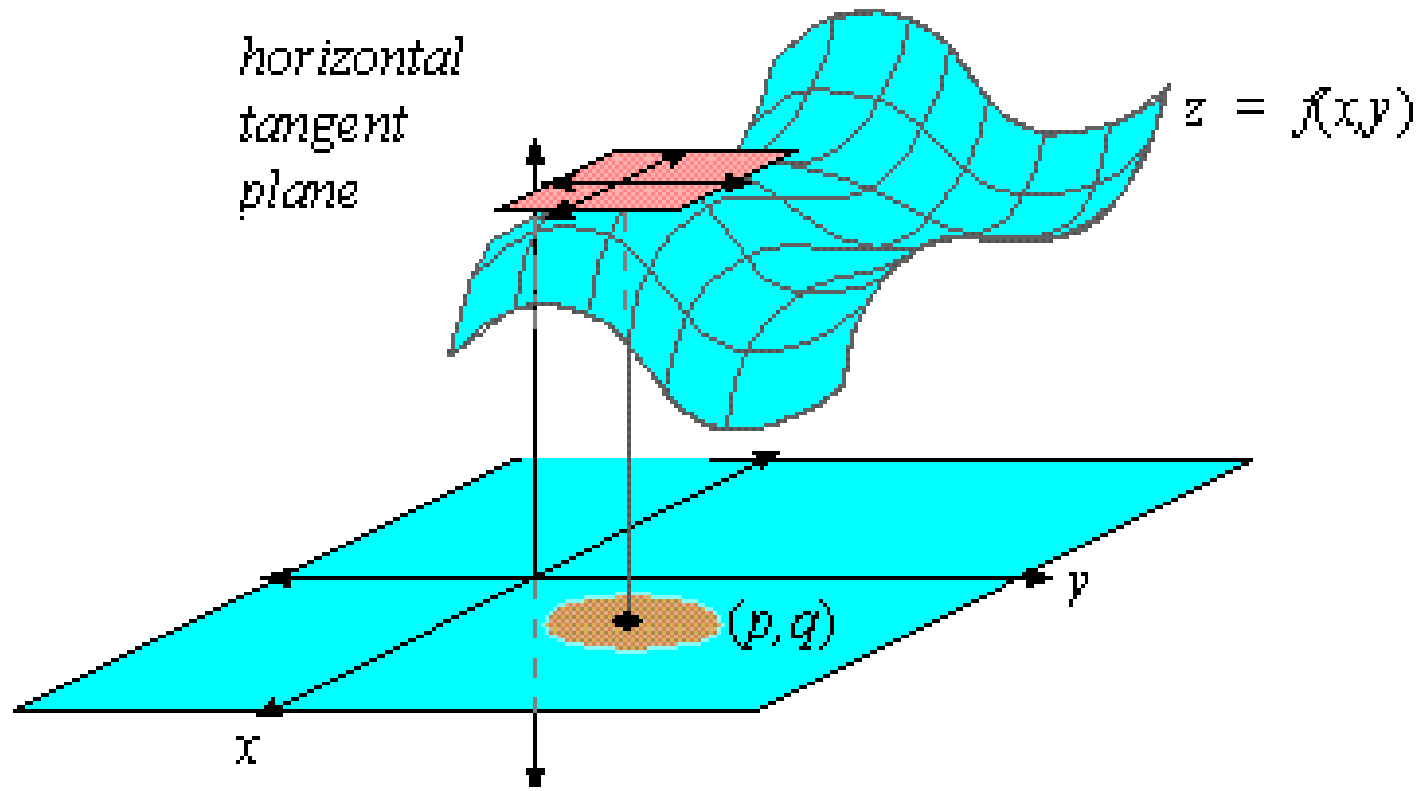
- Find all critical points
- Use the first or second derivative test to determine whether each point is a local maximum or minimum or neither.

# Functions of Two Variables: Critical Points

For a function  $z = f(x, y)$ , a critical point is where

$$f_x = f_y = 0 \quad \text{or} \quad f_x \text{ and/or } f_y \text{ does not exist.}$$

Note: *Both* partial derivatives must be zero (if they exist). This corresponds to the tangent plane being horizontal.



# Critical Points & Local Extrema

For a function  $z = f(x, y)$ ,

- If  $(a, b)$  is a local extremum then it is a *critical point*

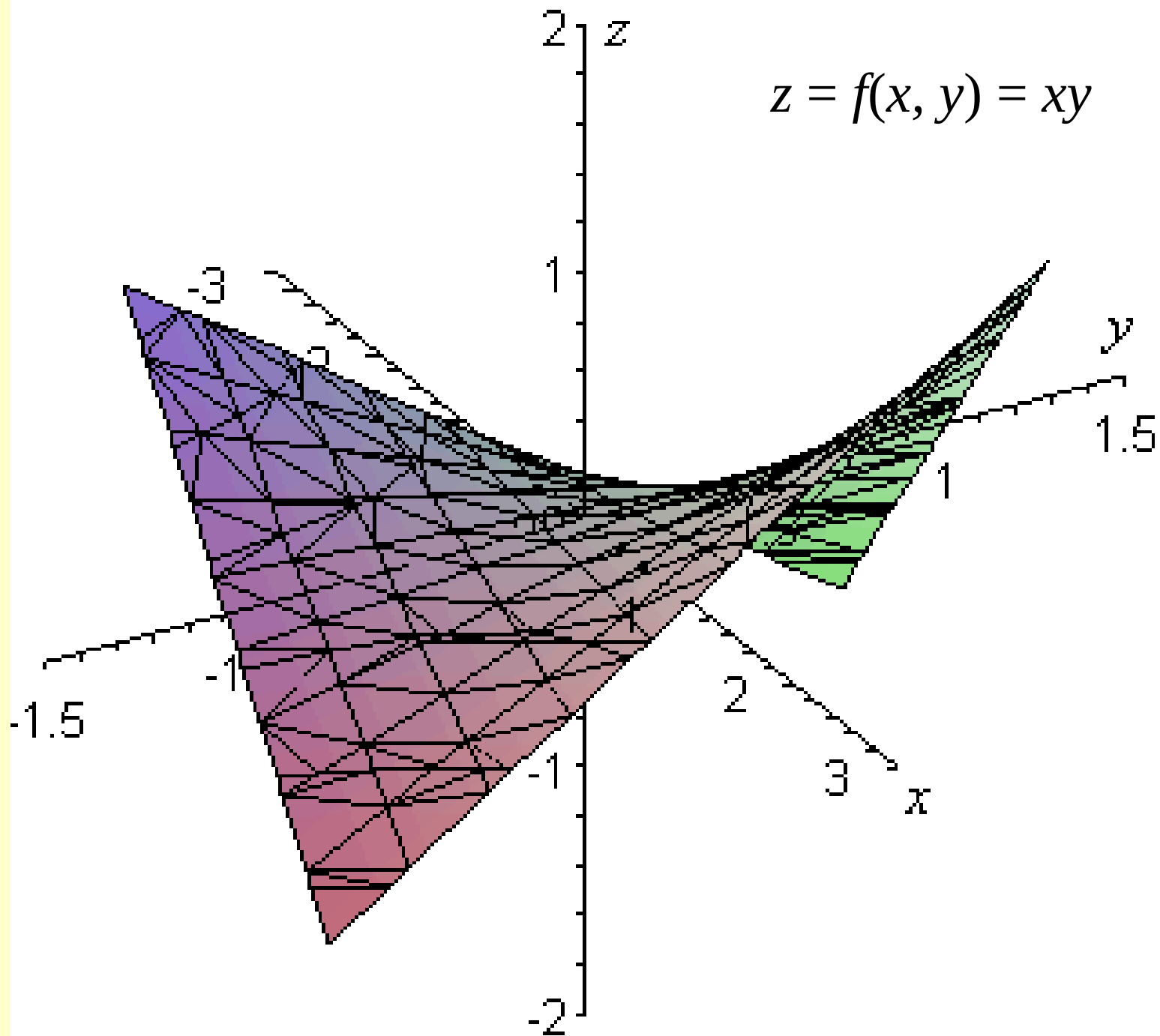
*Again:* Not all critical points are local extrema,  
but all local extrema are critical points!

**For example**, consider the function  $z = f(x, y) = xy$ .

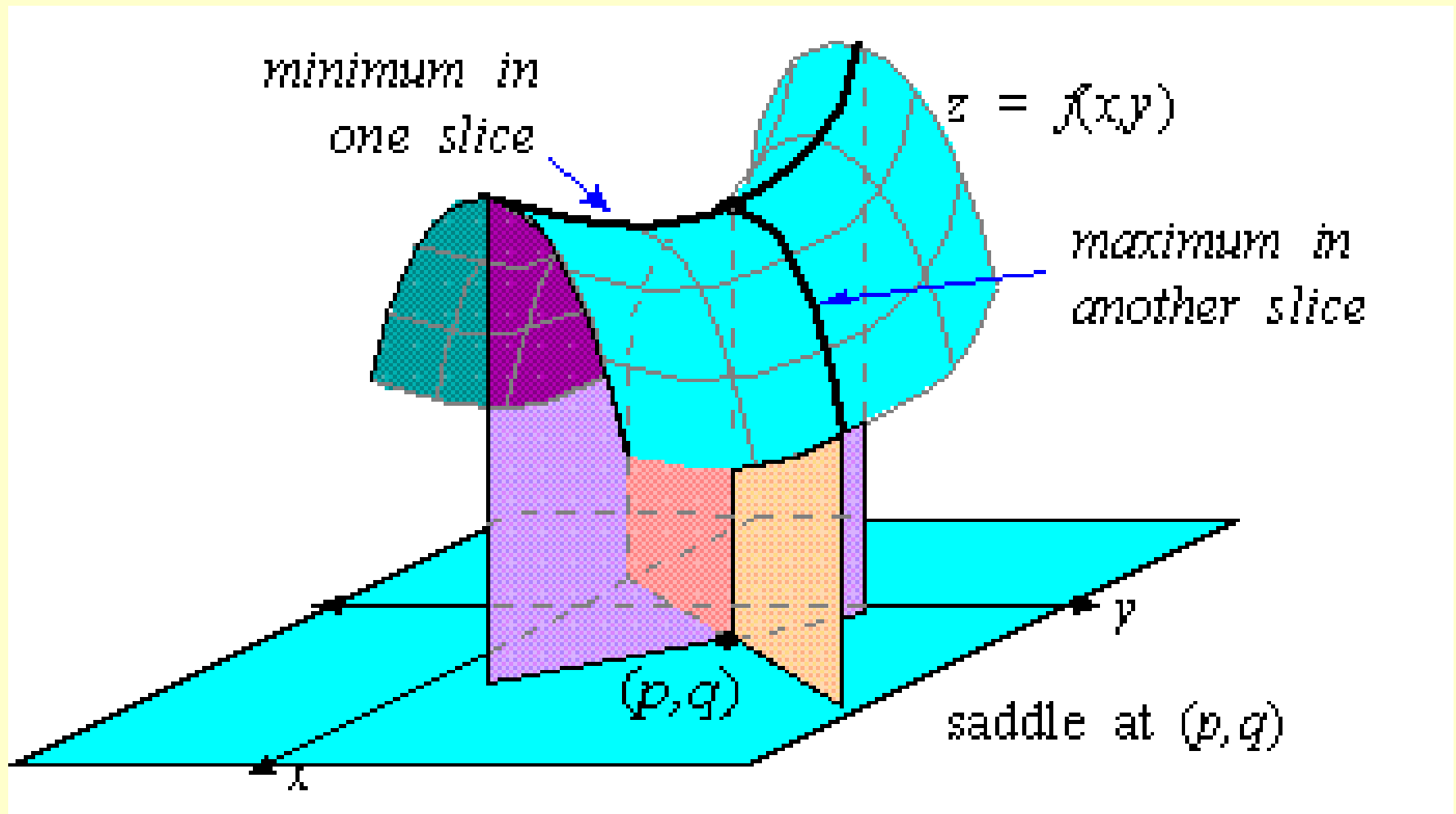
We have  $f_x(x, y) = y$  and  $f_y(x, y) = x$ .

So the function has one critical point: the point  $(0, 0)$ .  
By considering the equation or the graph it is clear  
that this point is not a local maximum or minimum.  
In fact it is called a **saddle point**.

$$z = f(x, y) = xy$$



# Saddle Point



## Example 18

Find the critical points of the function

$$f(x, y) = x^2 + y^2 + 2x - 4y + 10$$

### **Example 19**

Find the critical points of the function

$$z(x, y) = \frac{1}{3}x^3 - xy^2 - 2y$$

# Finding Local Extrema

To find the local extrema of a function  $z = f(x, y)$ ,

1. Find the critical points
2. Test the points to determine whether they are local maxima, minima or saddle points.

Assuming that the second order partial derivatives of  $f$  are all continuous in some region around the chosen critical point, we can use a **second derivative test**.

## Second Derivative Test

We define the ***discriminant*** of  $f(x, y)$  at point  $(a, b)$ :

$$D \equiv D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

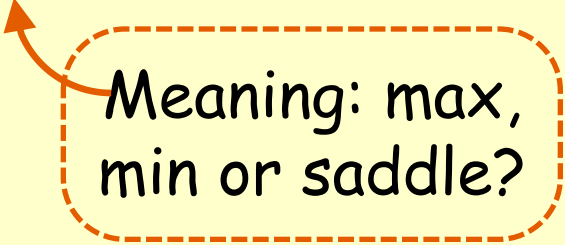
A critical point  $(a, b)$  can then be classified as follows:

1. If  $D > 0$  and  $f_{xx}(a, b) > 0$  then the point is a local minimum.
2. If  $D > 0$  and  $f_{xx}(a, b) < 0$  then the point is a local maximum.
3. If  $D < 0$  then the point is a saddle point.

Note If  $D = 0$ , this test is inconclusive and other tests must be used. Such cases are beyond the scope of this course.

## Example 20 (cf Example 18)

Classify the critical points of the function

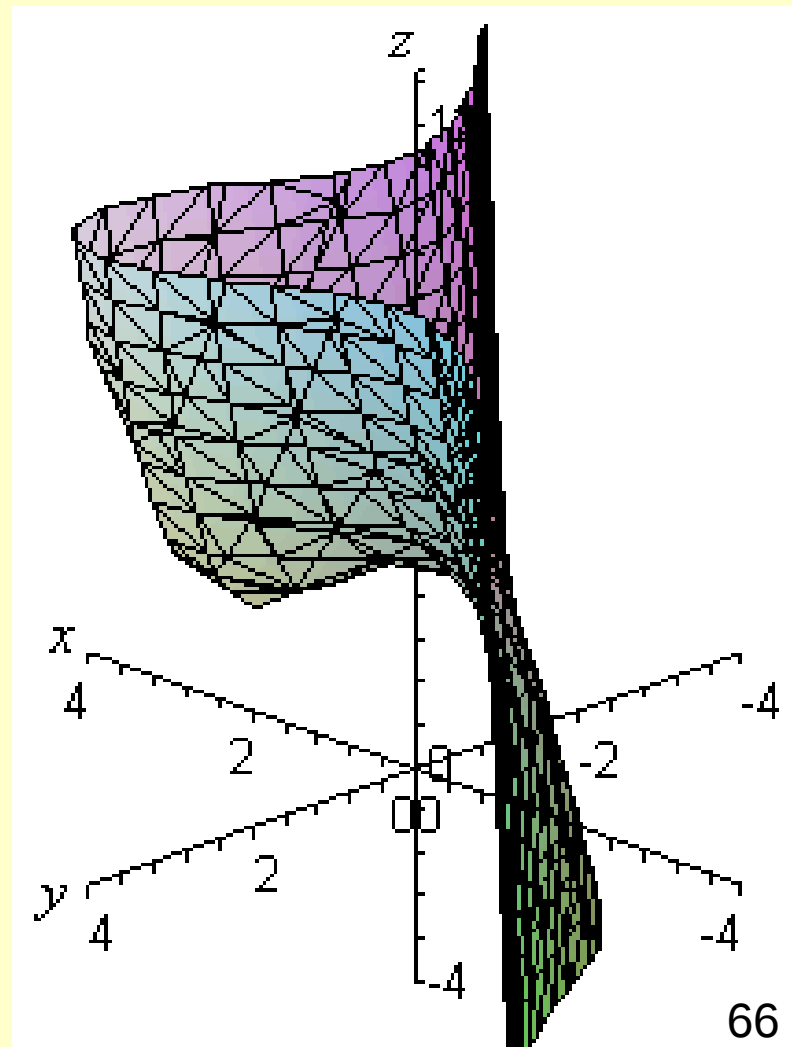


Meaning: max,  
min or saddle?

$$f(x, y) = x^2 + y^2 + 2x - 4y + 10$$

## Example 21

Find and classify all the critical points of the function  $z = f(x, y) = 4 + x^3 + y^3 - 3xy$



# Absolute Maxima and Minima: Definition

A function  $f(x, y)$  has an **absolute maximum** (or **global maximum**) at  $(a, b)$  if  $f(a, b) \geq f(x, y)$  for all points  $(x, y)$ .

A function  $f(x, y)$  has a **local minimum** (or **relative minimum**) at  $(a, b)$  if  $f(a, b) \leq f(x, y)$  for all points  $(x, y)$ .

# Functions of One Variable: Revision

Remember that for a function  $y = f(x)$ ,

- If  $f$  is continuous on a closed interval  $(a, b)$  then it has an absolute maximum and minimum on  $(a, b)$ .
- These absolute extrema occur *either* at critical points inside the interval *or* at the endpoints.

So to find absolute extrema on a closed interval, we:

1. Find all critical points
2. Evaluate the function at the critical points and at the end points.
3. The largest value found is the absolute maximum, the smallest value is the absolute minimum.

For functions of two variables, the ideas are similar ...

# Functions of Two Variables

- If a function  $z = f(x, y)$  is continuous on a closed, bounded set  $D$  in  $\mathbb{R}^2$  then it has an absolute maximum and minimum in  $D$ .
- These absolute extrema occur *either* at critical points inside  $D$  *or* on the boundary of  $D$ .

So to find the absolute extrema of  $f$  on  $D$  we:

1. Find all the critical points, and evaluate  $f$  at these points
2. Find the extreme values of  $f$  on the boundary of  $D$
3. The largest value of  $f$  found in (1) or (2) is the absolute maximum, the smallest value is the absolute minimum.

# Notes

- A “closed” set in  $\mathbb{R}^2$  is a set containing all its boundary points.
- A “bounded” set in  $\mathbb{R}^2$  is a set contained within some disc – i.e. it is finite in extent.

The main difference from the method for functions of a single variable is that instead of end points we have a *boundary*. I.e. we have one or more lines and need to find the extrema of the function on these lines, using methods from Calculus 1. This can take some time!

## Example 22

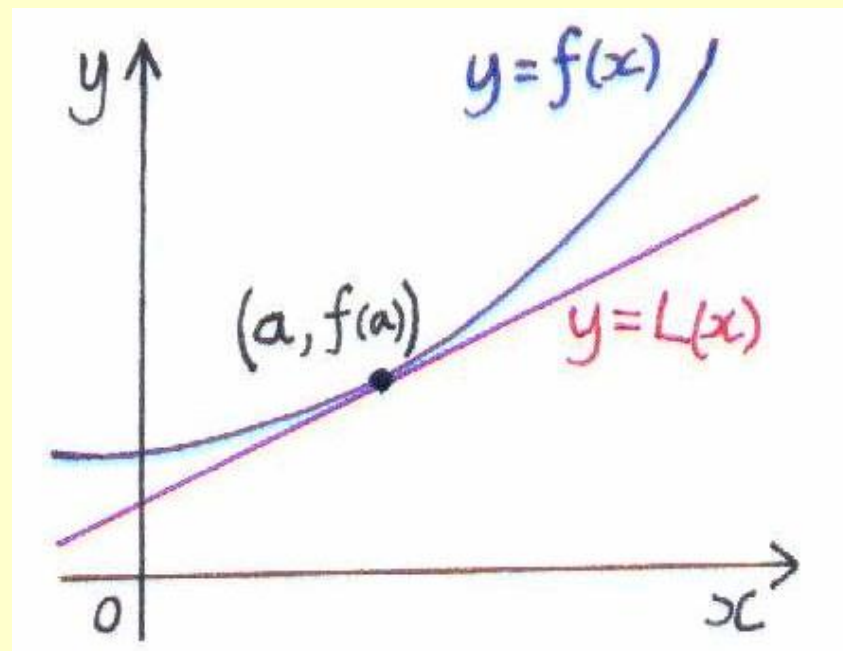
Find the absolute maximum and minimum values of the function  $z = f(x, y) = x^2 - 2xy + 2y$  on the rectangle  $D = \{(x, y) : 0 \leq x \leq 3, 0 \leq y \leq 2\}$

# **1.4 Total Differentials & Approximations**

## Linear Approximations: Revision

- The tangent to the curve  $y = f(x)$  at  $x = a$  has equation:  $L(x) = f(a) + f'(a)(x - a)$
- $L(x)$  is called the **linearisation** of  $f(x)$ .

• Near  $x = a$ , the curve lies very close to its tangent line, so we can use the tangent line as an approximation to the curve.



- The approximation  $f(x) \approx L(x) = f(a) + f'(a)(x - a)$  is called the **linear approximation** or **tangent line approximation** of  $f$  at  $a$ .

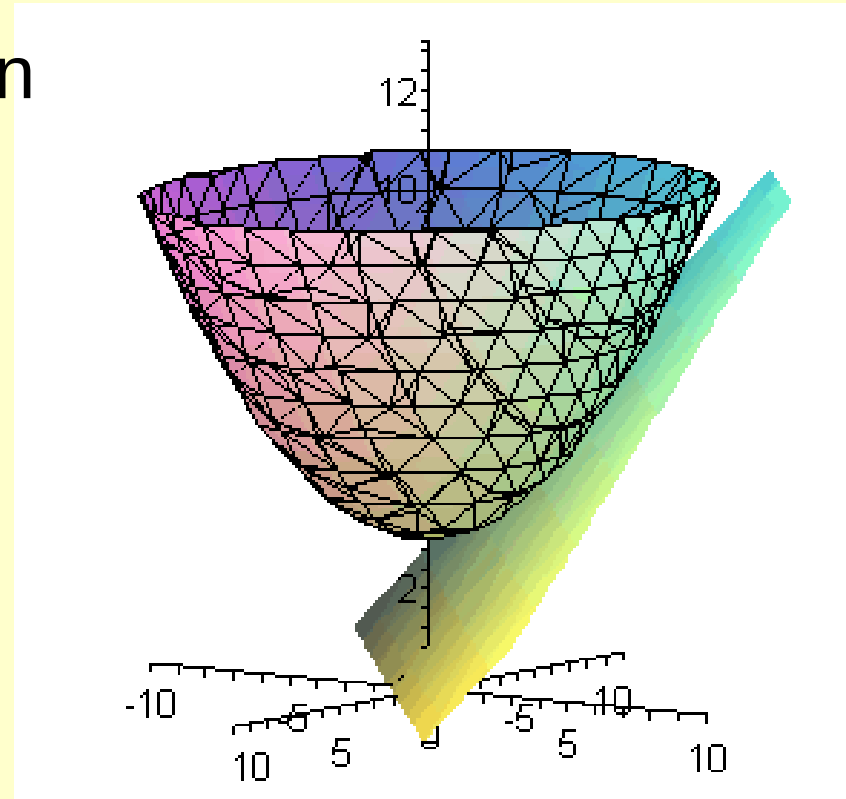
# Linear Approximations of Functions of Two Variables

- The linearisation of a function  $z = f(x, y)$  is:

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- This is actually the equation of the ***tangent plane*** to the surface  $f$  at the point  $(a, b)$ .

- It is the unique plane containing the tangent lines at  $(a, b)$  to the traces  $x = a$  and  $y = b$ .



- $f(x, y) \approx L(x, y)$  is the **linear approximation or tangent plane approximation** of  $f$  at  $(a, b)$ .

## Example 23

Find the linear approximation of the function

$$f(x, y) = 2x^2 + y^2 \text{ at } x = y = 1.$$

## Differentials: Revision

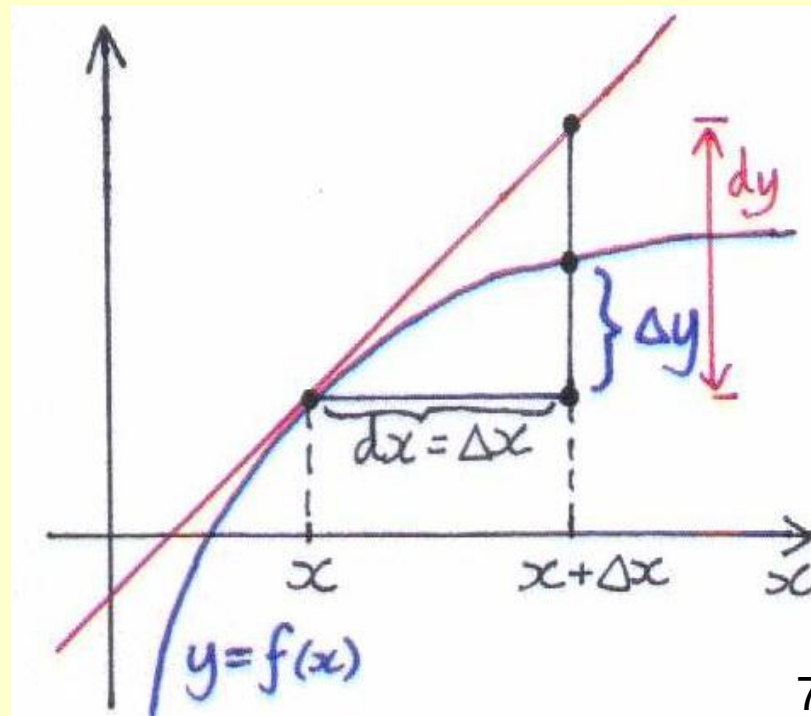
For  $y = f(x)$ , the **differential**  $dx$  is defined to be an independent variable: it can take any value.

The differential  $dy$  is then defined by  $dy = f'(x)dx$

Consider a change  $dx = \Delta x$  in  $x$ .

- $dy = f'(x)dx$  is the corresponding change in the *tangent* line.
- The corresponding change in the *function*,  $\Delta y$ , may be bigger or smaller than  $dy$ .

For small changes,  $\Delta y \approx dy$

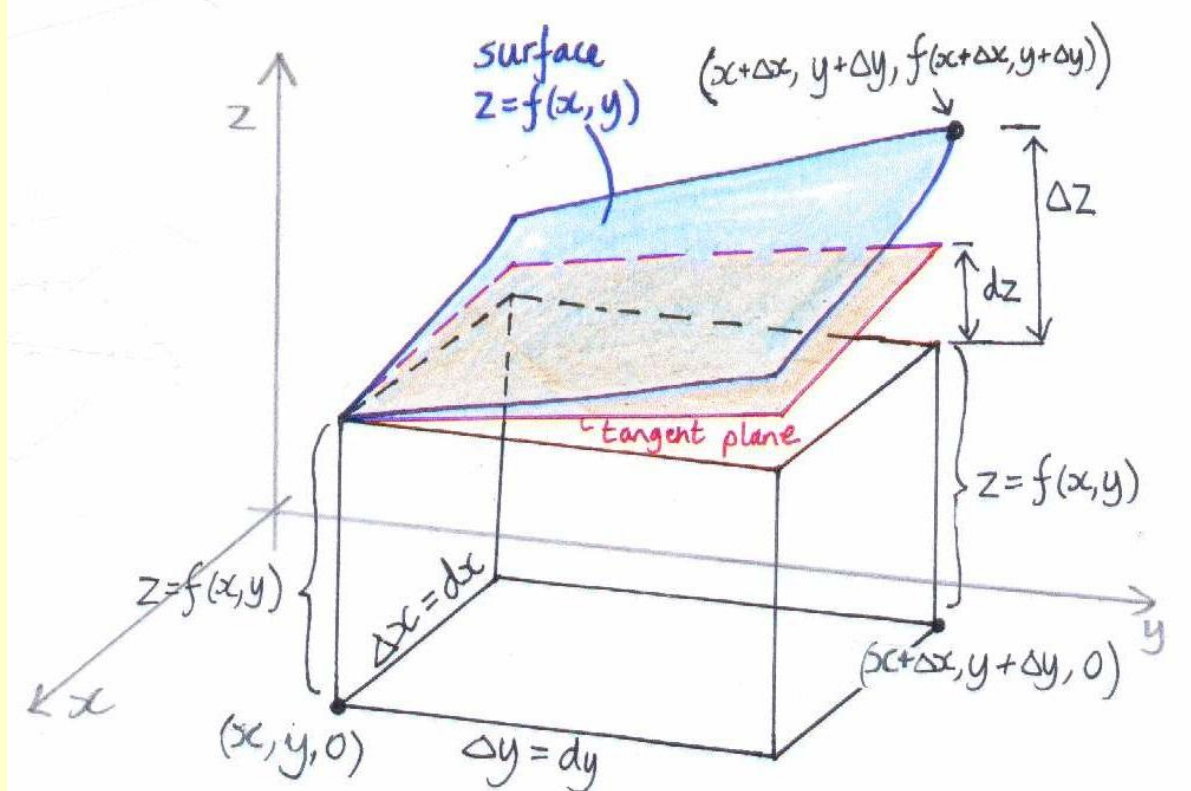


# Differentials of Functions of Two Variables

For  $z = f(x, y)$ , the **differentials**  $dx$  and  $dy$  are both defined to be independent variables.

The **(total) differential**  $dz$  is then defined by

$$dz = f_x(x, y) dx + f_y(x, y) dy = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$



## Example 24 (a)

- (a) If  $z = f(x, y) = 5x^2 + y + y^2$  find the differential  $dz$ .
- (b) If  $(x, y)$  changes from  $(1, 2)$  to  $(1.05, 2.1)$ , compare the values of  $\Delta z$  and  $dz$ .

One important application of differentials is in estimating errors.

## Revision

- Error = (estimated value) – (actual value)
- Often a more useful concept is **percentage error**:

$$\text{Percentage error} = \left| \frac{\text{error}}{\text{actual value}} \right| \times 100\%$$

### Example 24 (b)

A right circular cone is measured to have base radius  $(10.0 \pm 0.1)$  cm and height  $(25.0 \pm 0.1)$  cm. Find the volume of the cone and use differentials to find the maximum possible error and percentage error in the calculated volume.

# 1.5 Multiple Integrals

In Calculus 1 we studied definite integrals. These are related to the area under a curve. Now we will study integrals of functions of two variables, which are related to the volume under a surface.

## Revision: Area under a Curve

---

### Example from Calculus 1:

The table gives the values of a function  $f(x)$  on  $[0, 20]$ .

Estimate the value of the integral  $\int_0^{20} f(x) dx$  using

(a) left endpoints, (b) right endpoints, (c) midpoints.

|        |   |     |     |     |     |
|--------|---|-----|-----|-----|-----|
| $x$    | 0 | 5   | 10  | 15  | 20  |
| $f(x)$ | 1 | 1.7 | 2.2 | 1.9 | 1.6 |

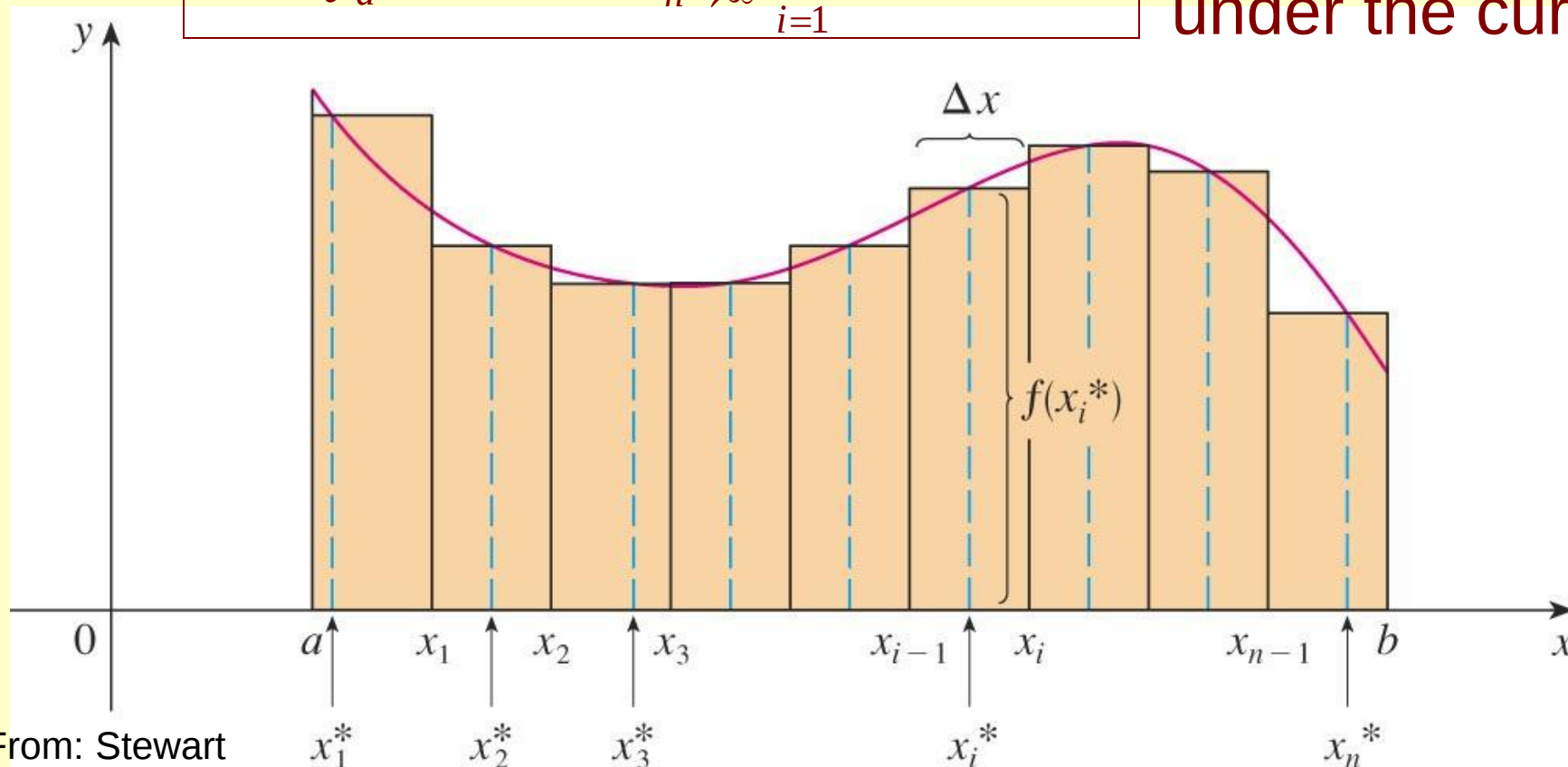
## Revision: The Definite Integral

- Suppose  $f(x)$  is a positive function on an interval  $[a, b]$ .
  - We divide  $[a, b]$  into  $n$  subintervals of width  $\Delta x = (b-a)/n$ .
- In each subinterval we choose a sample point  $x_i^*$ .

- Then

$$A = \int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

is the area under the curve.



## Volumes and Double Integrals

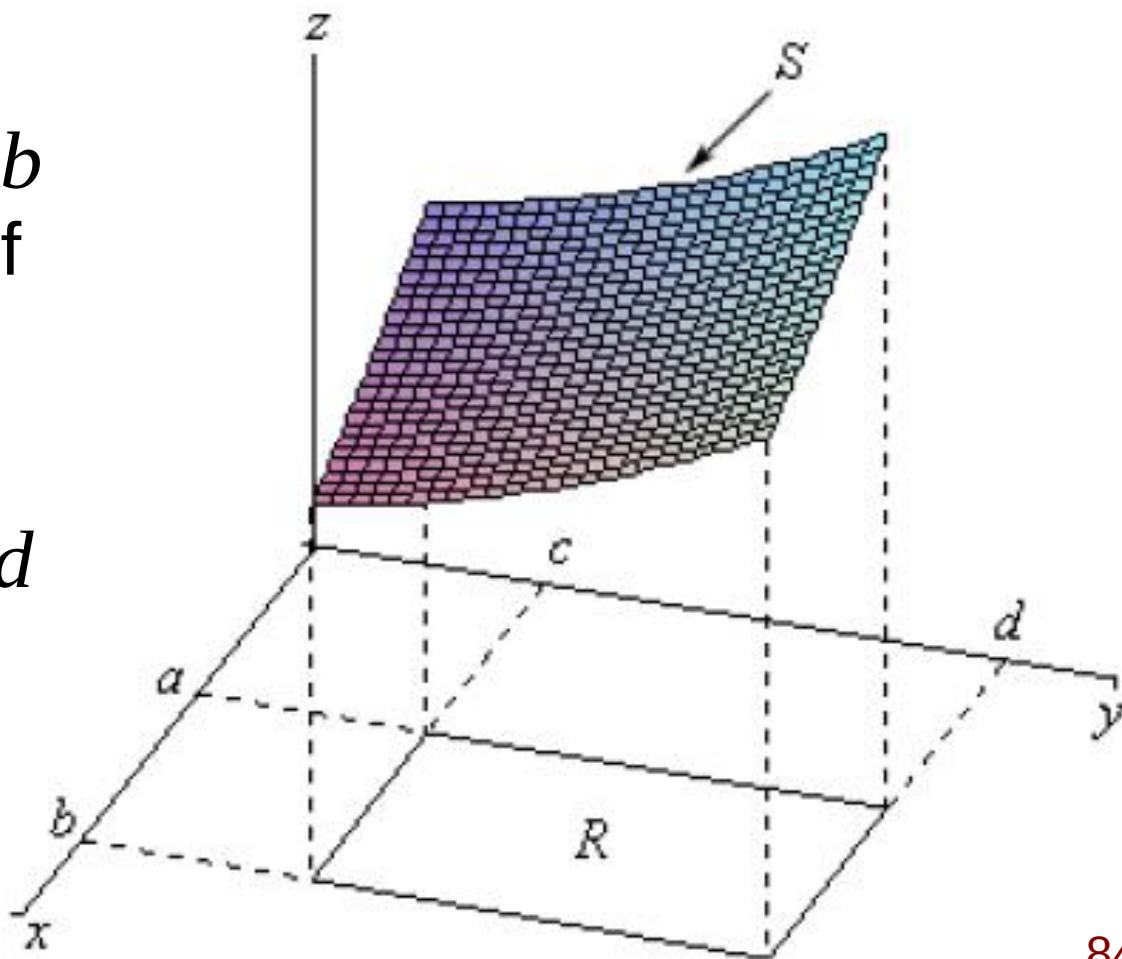
- Now consider a positive function  $z = f(x, y)$  on a closed rectangle,  $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$
- What is the volume of the solid lying under the surface  $f$  on  $R$ ?

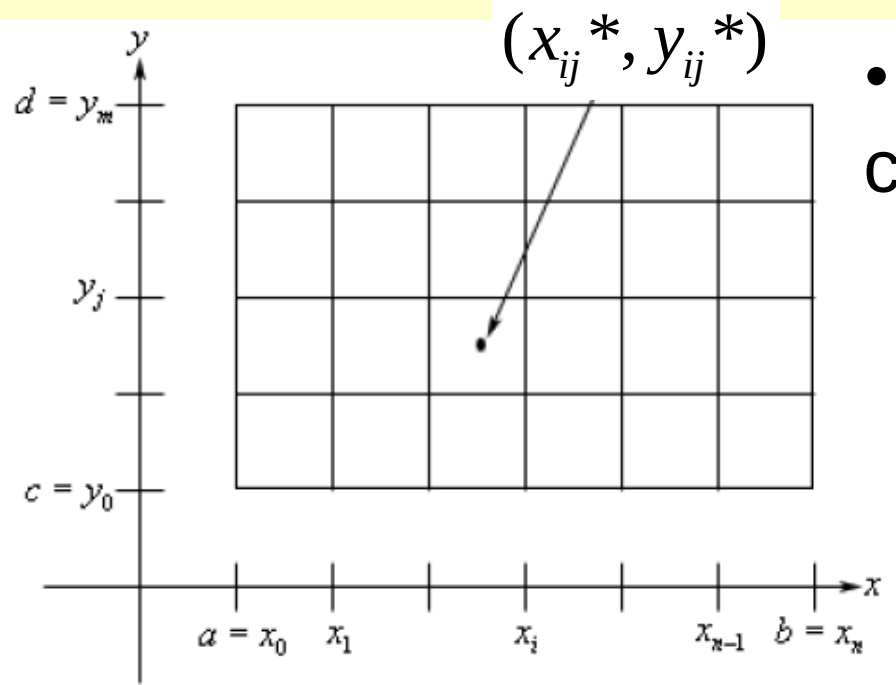
- We divide  $a \leq x \leq b$  into  $m$  subintervals of width

$$\Delta x = \frac{b - a}{m}$$

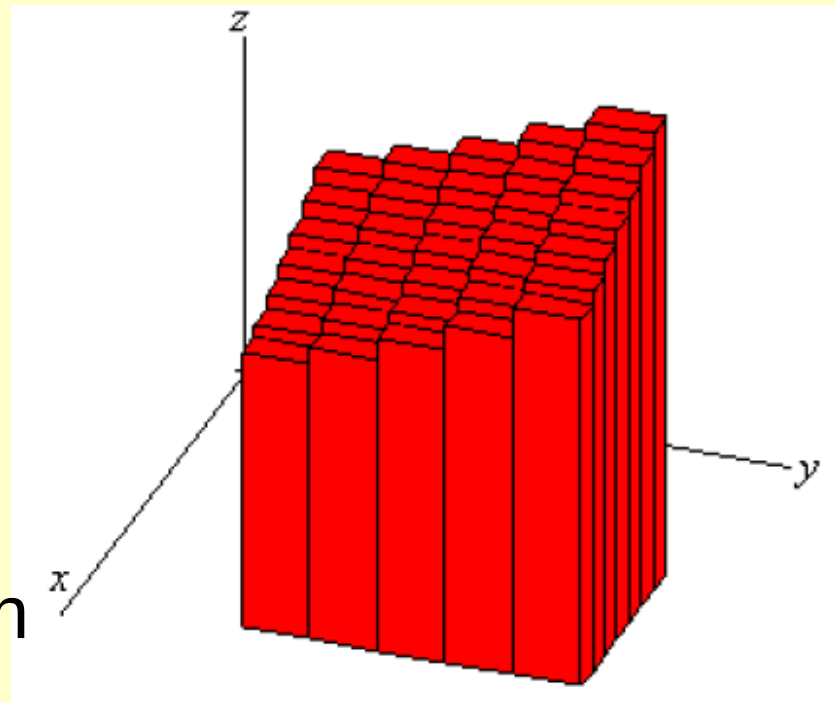
- We divide  $c \leq y \leq d$  into  $n$  subintervals

$$\Delta y = \frac{d - c}{n}$$





- In each sub-rectangle we choose a sample point  $(x_{ij}^*, y_{ij}^*)$



• The volume under surface  $f$  above each sub-rectangle can be approximated by a thin rectangular box with base area  $\Delta A = \Delta x \Delta y$  and height  $f(x_{ij}^*, y_{ij}^*)$ , giving volume  $f(x_{ij}^*, y_{ij}^*) \Delta A$ .

So the total volume is 
$$V \approx \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A.$$

$$V \approx \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A.$$

## Notes

1. The double sum means

- Evaluate  $f$  at the chosen point in each sub-rectangle then multiply by the area of the sub-rectangle
- Add the results for all rectangles.

2. Volumes can be estimated numerically using this idea. In particular, if we take the sample point to be

(a) at the **midpoint** of each sub-rectangle, notated

$(\bar{x}_i, \bar{y}_j)$ , we get: 
$$V \approx \sum_{i=1}^m \sum_{j=1}^n f(\bar{x}_i, \bar{y}_j) \Delta A.$$

(b) at the upper, right corner of each sub-rectangle,

$(x_i, y_j)$ , we get: 
$$V \approx \sum_{i=1}^m \sum_{j=1}^n f(x_i, y_j) \Delta A$$

Then we expect: 
$$V = \lim_{m,n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$$

It can be formally proved that this is the volume. Similar limits also occur in other situations. So in general, the **double integral** of  $f$  over the rectangle  $R$  is defined as

$$\iint_R f(x, y) dA = \lim_{m,n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$$

This represents the (signed) volume of the solid that lies under the surface  $f(x, y)$  above the region  $R$  in the  $xy$ -plane.

## Example 25

The table gives the values of a function  $f(x, y)$  defined on  $R = [1, 3] \times [0, 4]$ . Estimate  $\iint_R f(x, y) dA$  using

(a) midpoints,  $n = m = 2$ ,

(b) sample points furthest from the origin,  $n = m = 4$ .

| $x \backslash y$ | 0 | 1 | 2  | 3  | 4  |
|------------------|---|---|----|----|----|
| 1.0              | 2 | 0 | -3 | -6 | -5 |
| 1.5              | 3 | 1 | -4 | -8 | -6 |
| 2.0              | 4 | 3 | 0  | -5 | -8 |
| 2.5              | 5 | 5 | 3  | -1 | -4 |
| 3.0              | 7 | 8 | 6  | 3  | 0  |

# Iterated Integrals

- Evaluating double integrals directly from the definition is not easy!
- Many double integrals can be written as *iterated integrals* which can be found by evaluating two single integrals.

## Fubini's Theorem:

If  $z = f(x, y)$  is continuous on  $R = [a, b] \times [c, d]$  then

$$\begin{aligned}\iint_R f(x, y) dA &= \int_a^b \left( \int_c^d f(x, y) dy \right) dx \\ &= \int_c^d \left( \int_a^b f(x, y) dx \right) dy\end{aligned}$$

Note: only valid for **rectangular** regions!

These integrals are called *iterated integrals*.

First consider

$$\iint_R f(x, y) dA = \int_a^b \left( \int_c^d f(x, y) dy \right) dx$$

- 1)  $\int_c^d f(x, y) dy$  means
- $x$  is held fixed
  - $f$  is integrated with respect to  $y$  from  $y = c$  to  $y = d$

This procedure is called ***partial integration***.

The result is a function of  $x$ .

- 2) The second step is to integrate that function w.r.t.  $x$  from  $a$  to  $b$

Similarly,  $\int_c^d \left( \int_a^b f(x, y) dx \right) dy$  means integrate  $f$  w.r.t.  $x$  with  $y$  held fixed. This gives a function of  $y$  which is then integrated w.r.t.  $y$ .

## Example 26

Verify Fubini's theorem for  $f(x, y) = 3xy^2 + 1$  on  $R = [1, 3] \times [0, 2]$  by evaluating

$$(a) \int_1^3 \int_0^2 3xy^2 + 1 dy dx \qquad (b) \int_0^2 \int_1^3 3xy^2 + 1 dx dy$$

Note that there are two ways of doing these integrals. Sometimes one way is easier: choose carefully!

### Example 27

Evaluate the following integrals

$$(a) \iint_R \frac{x}{y} dA, \quad R = [0, 1] \times [1, 2]$$

$$(b) \iint_R x e^{xy} dA, \quad R = [-1, 2] \times [0, 1]$$

## Separable Integrals

Consider the **special case**  $f(x, y) = g(x)h(y)$ .

Then

$$\iint_R f(x, y) = \int_a^b \left( \int_c^d g(x)h(y) dy \right) dx$$

In finding the inner integral,  $x$  is treated as a constant so we can write

$$\iint_R f(x, y) = \int_a^b g(x) \left( \int_c^d h(y) dy \right) dx$$

The inner integral also yields a constant so we can write

$$\iint_R f(x, y) = \left( \int_a^b g(x) dx \right) \cdot \left( \int_c^d h(y) dy \right)$$

I.e. in this special case, the double integral can be written as the product of two single integrals.

### Example 28 (cf Example 27a)

Evaluate the following integral by writing it as the product of two single integrals:

$$\iint_R \frac{x}{y} dA, \quad R = [0,1] \times [1,2]$$

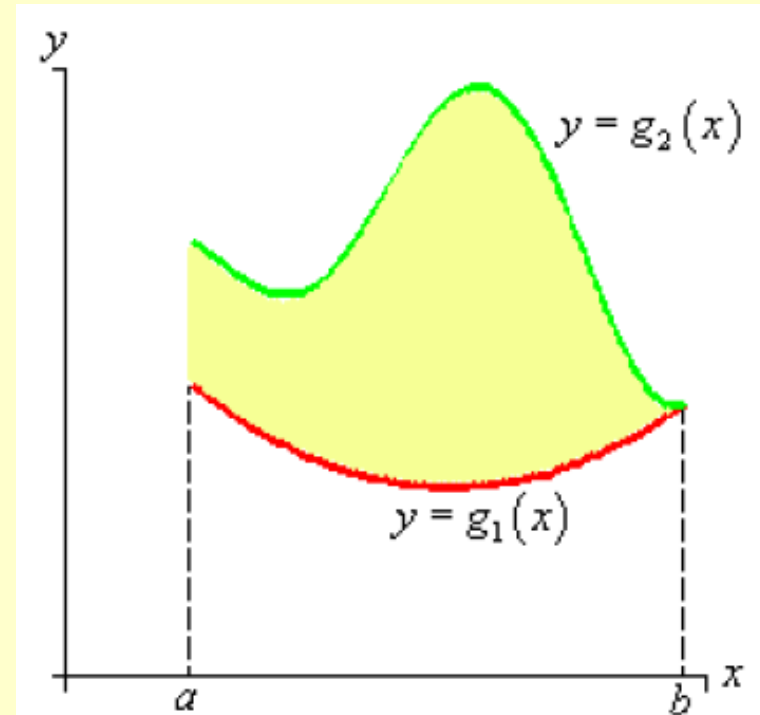
# Double Integrals over General Regions

Consider  $\iint_D f(x, y) dA$  where  $D$  is any (bounded) region on which  $f(x, y)$  is continuous. There are two basic types of region.

A **type I** region lies between the graphs of two functions of  $x$ :  $D = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$

Then

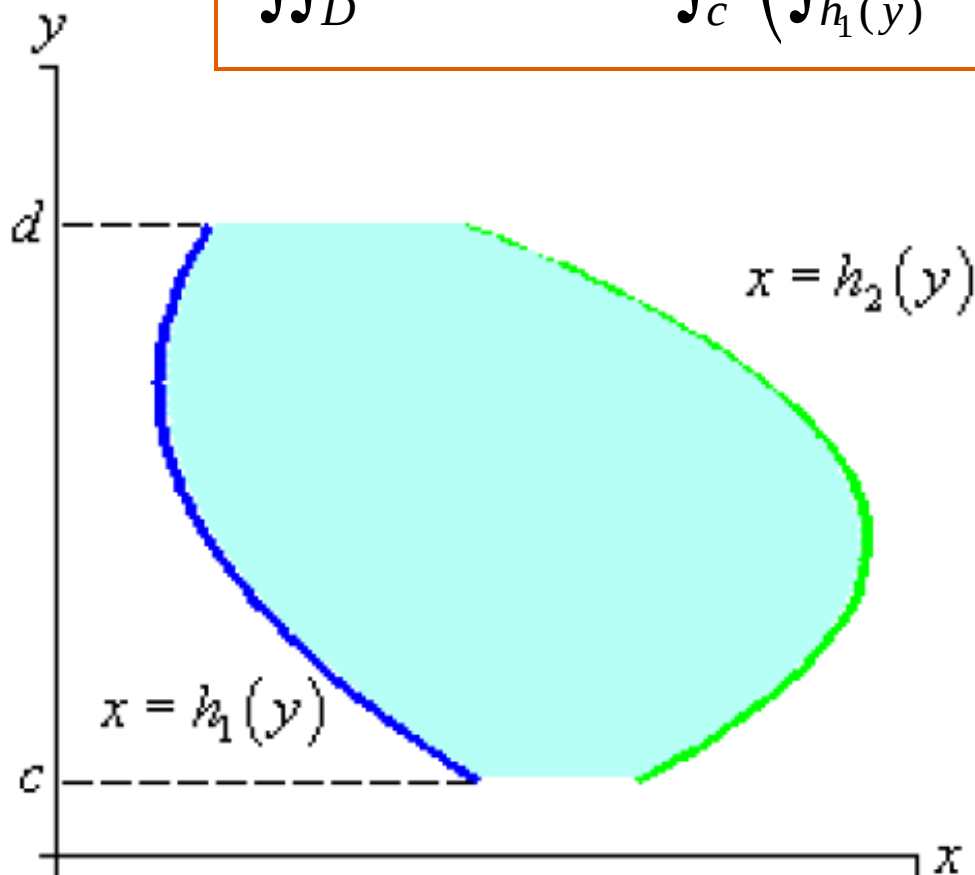
$$\begin{aligned} \iint_D f(x, y) dA \\ = \int_a^b \left( \int_{g_1(x)}^{g_2(x)} f(x, y) dy \right) dx \end{aligned}$$



A **type II** region lies between the graphs of two functions of  $y$ :  $D = \{(x, y) : h_1(y) \leq x \leq h_2(y), c \leq y \leq d\}$

Then

$$\iint_D f(x, y) = \int_c^d \left( \int_{h_1(y)}^{h_2(y)} f(x, y) dx \right) dy$$



**Example 29** Evaluate the following integrals:

(a)  $\iint_D e^{x/y} dA, \quad D = \{(x, y) \mid y \leq x \leq y^3, 1 \leq y \leq 2\}$

$$(b) \iint_D x^2 + \frac{1}{y^2} dA, \quad D = \{(x, y) \mid 1 \leq x \leq 2, \frac{1}{x} \leq y \leq x\}$$

(c)  $\iint_D (x + 2y) dA$  where  $D$  is the triangle with vertices at  $(0, 0)$ ,  $(1, 0)$  and  $(0, 1)$ .

(d) Find the volume under the surface  $f(x, y) = xy$  above the region of the  $xy$ -plane bounded by  $y = 0$ ,  $x = 1$  and  $y = x^2$ .

Some double integrals can be done in either of two ways, some are easier one way than the other, some can *only* be done one way. The order of integration can often be changed by sketching the region.

**Example 30**  $I = \int_0^1 \int_x^1 \sin(y^2) dy dx$

For the given integral, sketch the area of integration, then evaluate by changing the order of integration.

# Properties of Double Integrals

$$1. \iint_D [f(x, y) + g(x, y)] dA = \iint_D f(x, y) dA + \iint_D g(x, y) dA$$

$$2. \iint_D c f(x, y) dA = c \iint_D f(x, y) dA$$

3. If  $D = D_1 \cup D_2$  then

$$\iint_D f(x, y) dA = \iint_{D_1} f(x, y) dA + \iint_{D_2} f(x, y) dA$$

$$4. \iint_D dA = \text{area of region } D$$

# Triple Integrals

Consider a function  $f(x, y, z)$  continuous on a box

$$B = \{(x, y, z) : a \leq x \leq b, c \leq y \leq d, e \leq z \leq f\}$$

Similar to double integrals, we can define a triple integral

$$\iiint_B f(x, y, z) dV = \lim_{m, n, p \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V$$

Such integrals are similarly evaluated using “partial integration” – integrating with respect to one variable while holding the other *two* variables constant.

Integrals over general 3D regions can also be found – although establishing the appropriate limits in  $x$ ,  $y$  and  $z$  can be challenging!

**Example 31**

Evaluate  $\iiint_B \frac{x^2}{y} + e^z dV$  where

$$B = \{(x, y, z) : 0 \leq x \leq 1, 1 \leq y \leq e, 0 \leq z \leq \ln 2\}.$$